

Complementation in representable theories of region-based space

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Abstract

Through contact algebras we study theories of mereotopology in a uniform way that clearly separates mereological from topological concepts. We identify and axiomatize an important subclass of closure mereotopologies (CMT) called unique closure mereotopologies (UCMT) whose models always have orthocomplemented contact algebras (OCA) an algebraic counterpart. The notion of MT-representability, a weak form of spatial representability but stronger than topological representability, suffices to prove that spatially representable complete OCAs are pseudocomplemented and satisfy the Stone identity. Within the resulting class of contact algebras the strength of the algebraic complementation delineates two classes of mereotopology according to the key ontological choice between mereological and topological closure operations. All closure operations are defined mereologically iff the corresponding contact algebras are uniquely complemented while topological closure operations highly restrict the contact relation but allow not uniquely complemented and non-distributive contact algebras. Each class contains a single ontologically coherent theory that admits discrete models.

1 Introduction

Qualitative Spatial Reasoning (QSR) studies how the space surrounding us can be described using only qualitative aspects, i.e. without reference to some metric, and how we can efficiently automate reasoning with such descriptions. QSR has been a very active area of interdisciplinary research with interest from Artificial Intelligence (AI), Cognitive Science, Formal Logic, Geographical Information Systems, and Spatial Databases – just to name a few. Extensive introductions and overviews of QSR can be found, e.g., in [9; 10; 46]. Mereotopologies theories – which model only topological (of ‘connection’) and mereological (of ‘parthood’) aspects of space – are

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foundational within QSR. In the last two decades many first-order theories of mereotopology have been proposed for QSR, which has in turn led to fruitful systematic analyses exploring the ontological assumptions and the entailed logical properties of different sets of axioms for mereotopological theories [see 7; 19]. Closure Mereotopology (**CMT**: [7]) is widely accepted as the most restricted mereotopology that does not contain any controversial ontological assumptions. Though some specific extensions of **CMT** have been studied in great detail, the question of what constitutes a mereotopology that adequately represents physical space has been largely neglected. In particular, the existing work on specific mereotopologies suggests that still new combinations of axioms could yield yet unexplored theories of closure mereotopology. We give strong evidence why this is not the case. We do so by focusing on the spatial representability of the models of a mereotopology. Though many concrete embeddings of mereotopological models in topological spaces have been constructed [see 3; 11; 12; 13; 14; 15; 16; 44], the question of whether these topological representations adequately reflect the intended structure of physical space has not been addressed¹. As it turns out, the key in this pursuit is the necessary strength of the complementation operation. We show that assuming the existence of some kind of uniquely defined complements and requiring a weak form of spatial representability restrict the algebraic structure arising from mereotopologies to an extent that only a few particular theories remain. Only two distinct minimal classes of ontologically coherent mereotopologies (we define C-closure in that regard) are conceivable – distinguished by the presence or absence of unique complements. Our analysis further identifies the algebraic properties that correspond to the various closure operations and other ontological assumptions of the mereotopologies.

For our investigation we treat mereotopology algebraically as first proposed by [39; 41] and [15; 17]. The systematic studies of algebraic counterparts² of mereotopologies in [28; 44] offer many insights that help us understand the different mereotopological theories and the relationships amongst them. The study of algebraic theories of mereotopology is, for example, most convincing in separating the mereological component from the topological component as pointed out by [32]. We are particularly interested in *Unique Closure Mereotopology* (**UCMT**) and *Unique Infinitary Closure Mereotopology* (**UGMT**), subclasses of **CMT** of which *all* models have algebraic counterparts. The class **UCMT** includes many prominent mereotopologies such as the theories of Whitehead [47], of Clarke [8], the Region-Connection Calculus (**RCC**: [25; 36]), and its generalization (**GRCC**: [32]) which admits discrete models. The theory **UCMT** is introduced in Section 2; it assumes closure under (binary) sums and intersections just as **CMT** does, but additionally assumes closure under complementation with respect to a universal region and that all these closure operations are unique. In Section 3 we show that the algebraic counterparts of **UCMT** are orthocomplemented contact algebras (**OCA**). Thus, the spatial representability of **UCMT**s can be studied through the spatial representability of **OCA**s – a task we are much more comfortable with. In Section 4 we look at spatially representable **OCA**s, but in lack of a complete definition of spatial representability we resort to a weaker form thereof, **MT**-representability. We can show that every **MT**-representable complete **OCA** is pseudocomplemented and satisfies the Stone identity, i.e., is a **SPOCA**. For this result, we rely on the lattices being complete. However, this is only a minor restriction since we can reasonably expect all spatially representable

contact algebras to be complete. For discrete MT-representable mereotopologies, it is no restriction at all.

Section 5 contains our key contribution: We identify algebraic conditions that are necessary and sufficient for the closure operations sums, intersections, complements, and universal to be defined mereologically or topologically in SPOCAs. In particular, we show that the ontological choice between a mereological or topological complementation in a mereotopology is reflected in the algebraic structure: The algebras of mereologically closed mereotopological models are uniquely complemented and thus distributive while those of topologically closed models are only pseudo- and orthocomplemented but potentially non-distributive. This confirms how central complementation is in mereotopology as suggested by [40]. We identify the two minimal classes that emerge as MT-representable and ontology coherent (a notion defined later) algebraic structures from those two classes of SPOCAs in Section 6. The first class, namely weak Boolean contact algebras (WBCA), defines all closure operations mereologically; though only the more restricted generalized Boolean contact algebras (GBCA) are guaranteed to have intuitive spatial representations. The second class, namely SPOCAs with contact defined as $x\mathbf{C}y \Leftrightarrow x \not\leq y^\perp$, defines all closure operations topologically. These two classes are also the weakest ones that could axiomatize space as intended by Whitehead [47]. However, neither of them satisfies all conditions discussed by Whitehead. As a further consequence of our work, we can verify algebraically that the assumptions of Whiteheadian mereotopology as outlined in [20; 35] are not compatible with the connectivity axiom (Con) $\forall x\mathbf{C}(x, -x)$. Ways to overcome this problem are also discussed in Section 7. Furthermore, we prove that no ‘true mereotopology’, that is no MT-representable MT-closed mereotopology, with discrete models can exist. Only if we allow coherently closed (C-closed) instead of MT-closed mereotopologies, exactly two theories (amongst all combinations of mereological and topological definitions of each of the closure operation sum, intersection, complement, and universal), namely the GBCAs and the SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$, admit both continuous and discrete models.

On a different note, our work demonstrates that the duality between algebraic structures and topological spaces is not a mere theoretical exercise only of mathematical interest, but helps us understand the diversity of theories of qualitative space and select an axiomatization according to any given set of desirable ontological assumptions. Our methodology is outlined in Fig. 1: We leverage the knowledge about duality between certain lattices and topological spaces to the understanding of mereotopology. The models of all mereotopologies satisfying the discussed closure assumptions can be represented algebraically in a straightforward manner. With the introduced notion of MT-representability we are then able to reduce the contact algebras resulting from UCMTs to a much more restricted set of contact algebras, namely SPOCAs, that includes all spatially representable and ontologically coherent algebraic counterparts to models of UCMT. Two examples of such contact algebras arising from UCMTs, which are representative of the only two C-closed MT-representable contact algebras with discrete models, are given in Fig. 1. The figure also describes their logical counterparts as well as the common spatial interpretation of their models.

2 Mereotopologies with complements

We only consider so-called equidimensional mereotopologies, i.e. unsorted mereotopological theories whose domain elements can be interpreted as being of a single uniform dimension. For example, the domain elements could be interpreted all as 1D regions (such as time intervals or intervals on a line) as in Allen's Interval Algebra [1], or as spatial regions which are all 2D or all 3D, or as spatio-temporal regions of either all 3D or all 4D. Explicit multidimensional mereotopologies, i.e. mereotopologies with multiple sorts or explicit dimensions such as proposed in [21; 23; 24; 29], are beyond our scope here.

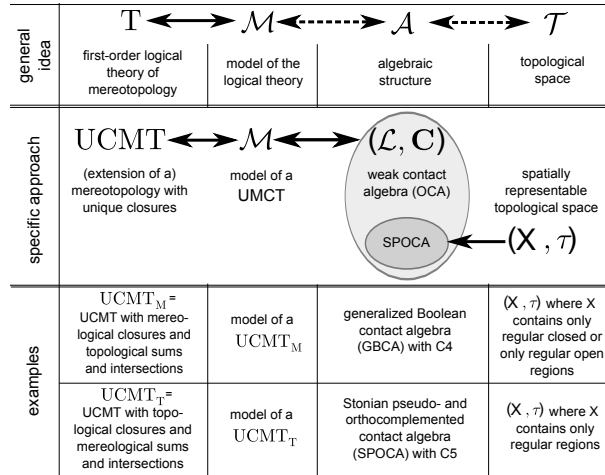


Figure 1 An overview of our approach. The correspondence between a logical theory of mereotopology and its model on the left-hand side are fairly standard. Representation results between some classes of algebraic structures, specifically lattices, and topological spaces as indicated on the right-hand side are known for some specific cases. In order to establish a subset of the logical theories of mereotopologies that have spatially representable topological interpretations, we need to, first and foremost, establish a correspondence between the models of **CMTs** and contact algebras. Since we cannot achieve this in general (indicated by the dashed arrow), we resort to the restriction of **CMTs** to **UCMTs** as shown in the second row. Every model of a **UCMT** has an algebraic counterpart in the class of orthocomplemented contact algebras (OCAs) as indicated by the solid arrow in the middle. As the second crucial step (the right dashed arrow in the top row), we try to reduce the class of OCAs to a smaller class that still includes all spatially representable contact algebras. This is a subclass of the Stonian pseudo- and orthocomplemented contact algebras (SPOCAs), which are at least MT-representable. However, as indicated by the uni-directed solid arrow on the right, not all SPOCAs are spatially representable or even topologically representable.

All equidimensional mereotopological theories consists of a single parthood and a single contact relation that satisfy the axioms (P.1)–(P.3) and (C.1)–(C.3) [45]. Such theories are commonly referred to as *ground mereotopologies* (**MT**) [7]. If C and/or P are not explicitly present or are not primitive relations, they still form an alternative set of primitives in a logically equivalent mereotopology. Throughout the paper we assume that any two regions with identical extensions of parthood *and* contact are identical. This follows immediately in our restriction to a single class of regions of equal dimensions.

Throughout the paper we assume standard first-order logic with equality and all logical sentences as implicitly universally quantified.

(P.1)	$P(x, x)$	(P reflexive)
(P.2)	$P(x, y) \wedge P(y, x) \rightarrow x = y$	(P anti-symmetric)
(P.3)	$P(x, y) \wedge P(y, z) \rightarrow P(x, z)$	(P transitive)
(C.1)	$C(x, x)$	(C reflexive)
(C.2)	$C(x, y) \rightarrow C(y, x)$	(C symmetric)
(C.3)	$C(z, x) \wedge P(x, y) \rightarrow C(y, z)$	(C monotone with respect to P)

Equivalent to (C.3) is the following axiom

$$P(x, y) \rightarrow \forall z(C(z, x) \rightarrow C(z, y))$$

Any such ground mereotopology allows defining the concepts of ‘overlap’ O, ‘underlap’ U, and ‘proper part’ PP in the following natural way:

(O)	$O(x, y) \leftrightarrow \exists z[P(z, x) \wedge P(z, y)]$	(Overlap)
(U)	$U(x, y) \leftrightarrow \exists z[P(x, z) \wedge P(y, z)]$	(Underlap)
(PP)	$PP(x, y) \leftrightarrow P(x, y) \wedge \neg P(y, x)$	(Proper Part)

In the sequel, we take these definitions for granted in any mereotopological theory.

2.1 Closure mereotopology with unique closures (UCMT) A common requirement for mereotopological theories are closure operations. These require an intersection for any two overlapping entities and a sum for any two underlapping entities, compare e.g. the closure mereotopology (**CMT**: [7]). Here, we go beyond **CMT** in three ways in order to define *unique closure mereotopology* (**UCMT**).

First, we require a greatest entity to exist, i.e. something that everything else is a part of (UCMT.4). The existence of such a universal entity is plausible in any restricted domain of interest, such as the earth, a specific country, building, or an even smaller experimental domain (such as a closed ‘blocks world’ consisting of a finite number of blocks).

$$(UCMT.4) \quad \forall x P(x, u) \quad (\text{Unique universal entity})$$

Secondly, we require all closure operations to be uniquely defined. The universal must be always unique by (P.2), we denote it by a constant u . If sums and intersections are unique for all pairs of entities, we can denote them by functions \oplus and \odot . Moreover, we need to ensure that the sum $x \oplus y$ is the smallest element which has both x and y as part (‘supremum’) and that anything that overlaps the sum must also overlap either x or y . Likewise, the intersection $x \odot y$ is the greatest element which is both part of x and y (‘infimum’). This is reflected in the axioms (UCMT.1) and (UCMT.2) which also ensure $P(x, x \oplus y)$ and $P(x \odot y, x)$. It follows that every pair of elements has a sum and intersection so that $P(x_1, x_2)$ and $P(y_1, y_2)$ together imply $P(x_1 \oplus y_1, x_2 \oplus y_2)$ and $P(x_1 \odot y_1, x_2 \odot y_2)$ if $O(x_1, y_1)$. In the presence of (UCMT.4) any two entities underlap, thus we do not require a conditional in (UCMT.1).

$$\begin{aligned} \text{(UCMT.1)} \quad & \forall z[(O(x,z) \vee O(y,z)) \leftrightarrow O(x \oplus y, z)] && \text{(Sum is supremum)} \\ \text{(UCMT.2)} \quad & O(x,y) \rightarrow \forall z[(P(z,x) \wedge P(z,y)) \leftrightarrow P(z, x \odot y)] && \text{(Intersection is infimum)} \end{aligned}$$

Finally, we require models not only to be closed under intersections and sums, but also to be closed under complementation – an additional primitive unary relation in the theory. Given that a universal entity exists, complements are a natural concept motivated by human perception of physical space: If we are given a restricted physical space, we can easily identify the complement with respect to the universal entity. Again, the complement shall be uniquely defined, hence we denote it by a function \ominus . Furthermore, the complement function shall be involutory (UCMT.5) – a reasonable assumption for uniquely defined complements. Additionally, (UCMT.6) and (UCMT.7) ensure that entities and their complement interact correctly with respect to sums and intersections (overlap). Though \ominus is a total function, the universal does not have a complement (cf. UCMT.3 further down). Thus, (UCMT.6) and (UCMT.7) do not apply to the universal u .

$$\begin{aligned} \text{(UCMT.5)} \quad & x = \ominus(\ominus x) && \text{(Complements involutory)} \\ \text{(UCMT.6)} \quad & x \neq u \rightarrow x \oplus (\ominus x) = u && \text{(Sum of complements)} \\ \text{(UCMT.7)} \quad & x \neq u \rightarrow \neg O(x, \ominus x) && \text{(Complements non-overlapping)} \end{aligned}$$

We do not restrict \oplus , \odot , and \ominus any further at this point. Instead, we consider in Section 5 two plausible definitions, a mereological and a topological one, of each of these functions.

Contrary to the existence of a universal entity, a null entity is cognitively undesirable. The null entity would be part of every entity, thus be in contact to every other entity, but on the other side not really existent, i.e. not in contact to anything at all. Therefore we postulate the following to ensure the cognitive adequacy of the mereotopological theories. However, it is not an essential assumption in our work because the algebraic counterparts of these mereotopologies explicitly introduce a null entity. Hence our analysis extends to mereotopologies with unique closures that allow or require a null entity such as [37].

$$\text{(UCMT.3)} \quad \forall x \exists y \neg P(x, y) \quad \text{(No null entity)}$$

Throughout this paper we use the term **UCMT** in the following broad sense:

Definition 2.1 Let $\mathfrak{M}\mathfrak{T}$ be a consistent, unsorted first-order theory with two distinguished binary predicates **C** and **P**, two binary functions \oplus , \odot , a unary function \ominus , and a constant u . If $\mathfrak{M}\mathfrak{T}$ entails the sentences (P.1)–(P.3), (C.1)–(C.3), and (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP), we call $\mathfrak{M}\mathfrak{T}$ a **UCMT**.

The domain elements in a model of **UCMT** are often called regions.

Any **UCMT** has a mereological component that is restricted to a closed mereology **CM** where sums, intersections, complements, and the universal are unique but is non-committal with respect to other mereotopological principles. These mereotopological principles, their corresponding axioms, and the properties of the resulting logical theories have been studied in much detail in [7; 19]. We will show later that the requirement of unique closures including unique complements does not leave many choices with respect to other mereotopological principles if we require spatial representability and ontological coherence.

2.2 General mereotopology with unique infinitary closures (UGMT) Many mereotopologies go beyond **CMT** by requiring sums and intersections of arbitrarily

many – possibly infinitely many – entities to exist. Axioms postulating such infinitary closures or unrestricted fusions either require axiom schemas or sets or classes, [cf. 7]. For better readability we use a set notation here; \mathcal{X} denotes an arbitrary set of domain entities:

$$(UGMT.1) \quad \forall \mathcal{X} [\forall z [\exists x \in \mathcal{X} (O(x, z)) \leftrightarrow O(\bigoplus \mathcal{X}, z)]] \quad (\text{Unrestricted sum})$$

$$(UGMT.2) \quad \forall \mathcal{X} [\exists z [\forall x \in \mathcal{X} (P(z, x)) \rightarrow \forall z [\forall x \in \mathcal{X} (P(z, x)) \leftrightarrow P(z, \odot \mathcal{X})]]] \\ (\text{Unrestricted intersections})$$

A **UCMT** that satisfies these axioms is a *general mereotopology* (**GMT**) with unique infinitary closures (including complements):

Definition 2.2 A **UGMT** is a **UCMT** that satisfies (UGMT.1) and (UGMT.2).

Only in Section 3.2 we will briefly discuss the subclass **UGMT** and how their algebraic counterparts yield complete lattices.

3 The algebraic structures arising from models of UCMT

We now introduce a class of algebraic structures called contact algebras and show that the models of **UCMT** correspond to orthocomplemented contact algebras (OCA) while the models of **UGMT** correspond to complete OCAs. First let us define what we mean by a contact algebra. Contact algebras are not a new concept, various classes thereof have been studied as algebraic counterparts of specific mereotopological theories, e.g. by [3; 15; 17; 39; 41; 44]. Our definition here encompasses the weakest common properties:

Definition 3.1 A contact algebra $(\mathcal{L}, \mathbf{C})$ consists of a bounded lattice \mathcal{L} which defines a partial order \leq and a contact relation \mathbf{C} that satisfies the following axioms:

$$(C0) \quad 0 - \mathbf{C}x \quad (\text{Null disconnectedness}) \\ (C1) \quad x \neq 0 \rightarrow x \mathbf{C}x \quad (\text{Reflexivity of } \mathbf{C}) \\ (C2) \quad x \mathbf{C}y \leftrightarrow y \mathbf{C}x \quad (\text{Symmetry of } \mathbf{C}) \\ (C3) \quad x \mathbf{C}y \wedge y \leq z \rightarrow x \mathbf{C}z \quad (\text{Monotonicity})$$

Thus, the contact relation must satisfy the axioms of a ground mereotopology. The axioms (C1)–(C3) are algebraic versions of the axioms (C.1)–(C.3) of **MTs** while (C0) deals with the newly introduced smallest element 0 that is necessary to construct a lattice from a mereotopological model. The assumption that 0 is not connected to any other entities is merely a convenient choice without deeper implications. To distinguish the contact relation in a mereotopological theory from the contact relation in its algebraic counterpart, we write $C(x, y)$ to refer to the former and $x \mathbf{C}y$ to refer to the latter.

3.1 Relevant classes of lattices Before we show how to construct the algebraic counterparts of **UCMTs**, we review the various classes of lattices necessary in our discussions throughout the remainder of the paper. These are used to define more restricted classes of contact algebras. Most of these classes of lattices are defined in standard references such as [6; 26], while more specialized classes are covered in [42]. Each class allows non-distributive models unless explicitly ruled out. The relations between these classes of bounded lattices are illustrated in Fig. 2.

One remark upfront: Any lattice can be treated as an algebraic structure $\langle L, \cdot, + \rangle$ as well as a partially ordered set $\langle L, \leq \rangle$ with unique supremum $+$ and unique infimum \cdot for any pairs of entities. We can define $x \leq y \leftrightarrow x \cdot y = x$ for any $x, y \in L$. Throughout

Boolean lattice = relatively unicompl. = pseudocompl. section-semicompl.

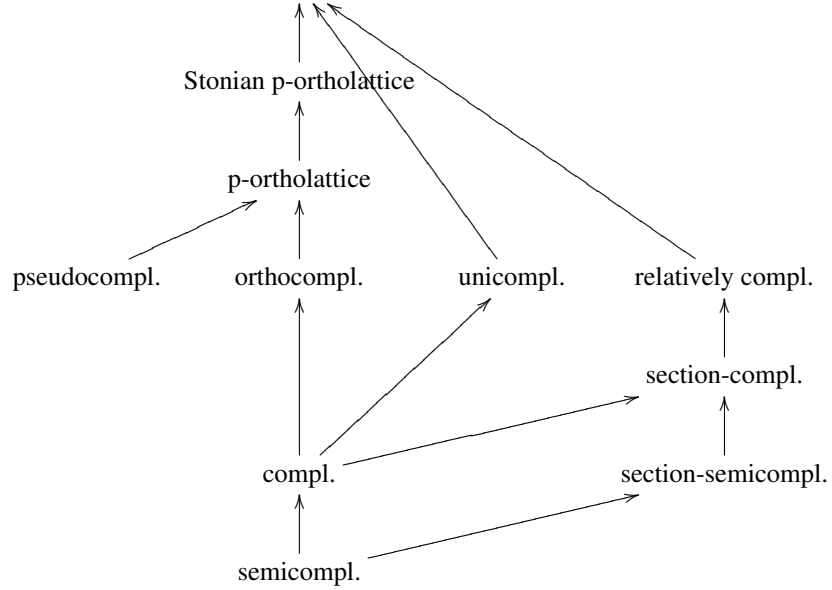


Figure 2 Relationships between bounded lattices with varying kinds of complementation; adapted from [27; 42]. The arrows indicate refinement, e.g. every p-ortholattice is also a pseudocomplemented and an orthocomplemented lattice. These refinements are transitive. In the case of distributive bounded lattices many of these classes of lattices collapse.

the paper, we depict lattices as Hasse diagrams which are transitive reductions of the partial order of the lattice. That means only the direct, i.e. covering, order relations are depicted while transitive closure is implied. $x \leq y$ holds if and only if there is a path consisting of one or multiple line segments strictly leading upwards from x to y .

Definition 3.2 A bounded lattice is a structure $\langle L, \cdot, +, 0, 1 \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ such that

- (B0) $\langle L, \cdot, + \rangle$ is a lattice, i.e. $a + b$ and $a \cdot b$ are uniquely defined for all $a, b \in L$;
- (B1) there exists an element $1 \in L$ so that $1 \cdot a = a$ (and $1 + a = 1$) for all $a \in L$;
- (B2) there exists an element $0 \in L$ so that $0 \cdot a = 0$ (and $0 + a = a$) for all $a \in L$.

Definition 3.3 A bounded distributive lattice is a structure $\langle L, \cdot, +, 0, 1 \rangle$ such that

- (D0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice,
- (D1) the distributive law holds, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in L$.

The structures in the following Definitions 3.4 to 3.8 are all of type $\langle 2, 2, 1, 0, 0 \rangle$ equipped with a unary function of complementation or pseudocomplementation.

Definition 3.4 A complemented lattice is a structure $\langle L, \cdot, +, ', 0, 1 \rangle$ such that

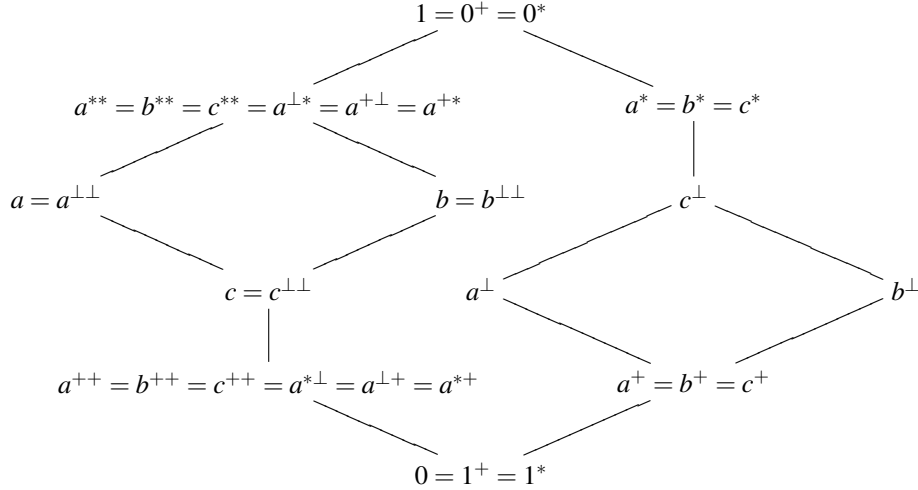


Figure 3 A simple p-ortholattice with orthocomplements, pseudo-complements, and quasicomplements indicated.

- (O0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice,
(O1) a' is a complement of a , i.e. $a + a' = 1$ and $a \cdot a' = 0$.

Definition 3.5 An orthocomplemented lattice (short: ortholattice) is a structure $\langle L, \cdot, +, \perp, 0, 1 \rangle$ such that

- (O0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice,
(O1) a^\perp is an orthocomplement of a , i.e. for all $a, b \in L$ we have
(a) $a^{\perp\perp} = a$,
(b) $a \cdot a^\perp = 0$,
(c) $a \leq b$ implies $b^\perp \leq a^\perp$.

Notice that orthocomplemented lattices are complemented.

Definition 3.6 A pseudocomplemented lattice is a structure $\langle L, \cdot, +, *, 0, 1 \rangle$ such that

- (P0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice,
(P1) a^* is the pseudocomplement of a , i.e. for all $b \in L$, $a \cdot b = 0 \iff b \leq a^*$.

Definition 3.7 A quasicomplemented lattice is a structure $\langle L, \cdot, +, ^+, 0, 1 \rangle$ such that

- (Q0) $\langle L, \cdot, +, 0, 1 \rangle$ is a bounded lattice,
(Q1) a^+ is the quasicomplement of a , i.e. for all $b \in L$, $a + b = 1 \iff b \geq a^+$.

Quasicomplemented lattices are also known as dually pseudocomplemented lattices.

Definition 3.8 A uniquely complemented lattice (short: unicomplemented lattice) is a structure $\langle L, \cdot, +, ', 0, 1 \rangle$ such that

- (U0) $\langle L, \cdot, +, ', 0, 1 \rangle$ is a complemented lattice,
(U1) a' is the unique complement of a , i.e. for all $b \in L$, $b + a = 1$ and $b \cdot a = 0$ imply $b = a'$.

Clearly, every uniquely complemented lattice is orthocomplemented, but not necessarily pseudocomplemented or quasicomplemented. On the other side, Fig. 3 gives an example of a orthocomplemented, pseudocomplemented, and quasicomplemented lattice which is not unicomplemented. Pseudo- or quasicomplemented lattices do not even have to be complemented. Lattices that are both pseudocomplemented and orthocomplemented (and thus also complemented and quasicomplemented) but not unicomplemented were introduced in [30] as p-ortholattices.

Definition 3.9 A p-ortholattice is a structure $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ such that

- (PO0) $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ is a quasicomplemented lattice,
- (PO1) $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ is an ortholattice.

An ortholattice is pseudocomplemented if and only if it is quasicomplemented. For a given p-ortholattice $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$, the structure $\langle L, \cdot, +, *, 0, 1 \rangle$ is a pseudocomplemented lattice if we define $x^* = x^{\perp\perp}$. P-ortholattices in which the orthocomplementation and pseudocomplementation operations coincide (unlike Fig. 3) are unicomplemented. Unicomplemented ortholattices are Boolean [5].

Definition 3.10 A Boolean lattice is a structure $\langle L, \cdot, +, ', 0, 1 \rangle$ such that

- (BO0) $\langle L, \cdot, +, ', 0, 1 \rangle$ is an orthocomplemented lattice,
- (BO1) the distributive law holds, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in L$.

But there are other interesting subclasses of p-ortholattices that are not distributive and thus not Boolean. Stonian p-ortholattices were introduced in [30] to algebraically capture the structure of the mereotopology of [2]. A Stonian p-ortholattice is a p-ortholattice that satisfies the Stone identity (SPO1). Fig. 4 illustrates that not all p-ortholattices are Stonian.

Definition 3.11 A Stonian p-ortholattice is a structure $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ such that

- (SPO0) $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ is a p-ortholattice,
- (SPO1) The Stone identity holds, i.e. $(a + b)^+ = a^+ \cdot b^+$ for all $a, b \in L$.

Again, a Stonian p-ortholattice $\langle L, \cdot, +, ^\perp, 0, 1 \rangle$ can also be defined equivalently as $\langle L, \cdot, +, *, ^\perp, 0, 1 \rangle$ using the pseudocomplementation operation if we choose $x^* = x^{\perp\perp}$. We use both structures interchangeably. The Stone identity was originally proposed by Marshall Stone as an immediate generalization of Boolean algebras to so-called Stone lattices – pseudocomplemented distributive lattices which satisfy the Stone identity. Several other ways of stating the Stone identity are known, amongst them $a^* + a^{**} = 1$ and $(a \cdot b)^{++} = a^{++} \cdot b^{++}$. But one version of the Stone identity, that is in distributive lattices also equivalent to those, namely $a^* + a^{**} = 1$, is inadequate here since it holds for all p-ortholattices. Stonian p-ortholattices generalize the (distributive) Stone lattices to non-distributive lattices.

Notice that the dual of (SPO1), $(a \cdot b)^+ = a^+ + b^+$, holds for all quasicomplemented lattices and, equally, $(a + b)^* = a^* \cdot b^*$ holds for all pseudocomplemented lattices. Moreover, (SPO1) and its dual hold for orthocomplements in ortholattices, that is $(a + b)^\perp = a^\perp \cdot b^\perp$ and $(a \cdot b)^\perp = a^\perp + b^\perp$ for all $a, b \in L$ if L is orthocomplemented [30]. Finally, it is easily verifiable that that Boolean lattices are Stonian p-ortholattices.

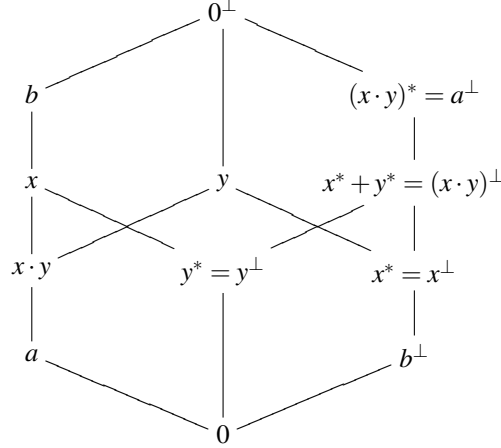


Figure 4 A p-ortholattice that violates $(x \cdot y)^* = x^* + y^*$ and is therefore not Stonian [30].

3.2 Orthocomplemented contact algebras (OCA) We now show that all the models of a UCMT can be viewed algebraically as contact algebras in which the lattice is orthocomplemented³.

Definition 3.12 An orthocomplemented contact algebra (OCA) is an algebraic structure $\mathcal{A} = (\mathcal{L}, \mathbf{C})$ consisting of an ortholattice $\mathcal{L} = \langle L, +, \cdot, ', 0, 1 \rangle$ equipped with a contact relation \mathbf{C} satisfying (C0) to (C3).

The theory $T_{OCA} = \{L2^\vee-L6^\vee, L2^\wedge-L4^\wedge, O1'-O3', C0-C3\}$ axiomatizes OCAs, see Appendix A for the axioms. Notice that OCAs are not necessarily distributive. We only consider non-trivial OCAs which contain an element apart from 0 and 1. Now we show how to construct an OCA from an arbitrary model of UCMT.

Theorem 3.1 Every model \mathcal{M} of UCMT is homomorphic to some OCA.

Proof Define the mapping g from \mathcal{M} with domain M into the algebraic structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle L, +, \cdot, ', 0, 1 \rangle, \mathbf{C})$ with $L = M \cup \{0\}$ and $0 \notin M$ as following:

1. $g(x) = x$;
2. $g(\ominus x) = x^\perp$ for all $x \neq u$;
3. $g(x \odot y) = x \cdot y$ iff $x \mathbf{O} y$;
4. $g(x \oplus y) = x + y$;
5. $g(u) = 1$;
6. $x \mathbf{C} y$ iff $C(x, y)$;
7. $1^\perp = 0$ and $0^\perp = 1$;
8. $g(x \odot y) = 0$ if $x - \mathbf{O} y$;
9. $0 + x = x$ and $0 \cdot x = 0$;
10. $0 - \mathbf{C} x$.

We now need to show that

- (i) g is a homomorphism (structure preserving),
- (ii) \mathcal{L} is an ortholattice, and
- (iii) \mathbf{C} satisfies (C0) to (C3).

(i): It is easy to see that g is an injective function. It is a homomorphism because the operations \ominus, \odot, \oplus , directly correspond to $^\perp, \cdot, +$ for all elements in M .

(ii): Since \oplus and \odot define supremum and infimum for every pair of elements (infimum is defined as 0 for all non-overlapping pairs), $\mathcal{L} = (M \cup \{0\}, +, \cdot, ', 0, 1)$ is a lattice with $x \leq y \leftrightarrow x \cdot y = x$ and thus:

$$(11) \quad x \leq y \iff P(x, y) \text{ or } x = 0$$

By (8) and (9), the lattice has in 0 a lower bound. By (5) and (UCMT.4), it has in 1 an upper bound. Thus, \mathcal{L} is a bounded lattice.

\mathcal{L} further satisfies the properties O1(a) to O1(c) of ortholattices:

O1(a): follows because $^\perp$ is involutory (by UCMT.5) and (7).

O1(b): follows from $x \cdot x^\perp = 0$ by (UCMT.7) and (8).

O1(c): by (11) and (2) it suffices to prove $P(x, y) \rightarrow P(\ominus y, \ominus x)$. For $x = 0$ or $y = 0$, it holds trivially by (7), (9), (5), and (UCMT.4). Now suppose for two elements $x, y \in M \setminus \{0\}$, $P(x, y)$ but not $P(\ominus y, \ominus x)$. Then we get a contradiction from the following derivation:

$$\begin{aligned} & \exists z [P(z, \ominus y) \wedge \neg P(z, \ominus x)] && \text{P transitive and anti-symmetric (P.2), (P.3)} \\ \Rightarrow & \exists z [P(z, \ominus y) \wedge O(z, x)] && \text{(UCMT.1), (UCMT.6)} \\ \Rightarrow & \exists z, v [P(z, \ominus y) \wedge P(v, z) \wedge P(v, x)] && \text{definition of O (O)} \\ \Rightarrow & \exists v [P(v, \ominus y) \wedge P(v, x)] && \text{P transitive (P.3)} \\ \Rightarrow & \exists v [\neg P(v, y) \wedge P(v, x)] && \text{(UCMT.7), definition of O (O)} \\ \Rightarrow & \neg P(x, y) && \text{P transitive (P.3)} \end{aligned}$$

(iii): The contact relation \mathbf{C} satisfies (C0) by definition and (C1)–(C3) follow directly from (C.1)–(C.3) of a UCMT.

Thus, the structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle M \cup \{0\}, +, \cdot, ', 0, 1 \rangle, \mathbf{C})$ is an OCA and g is a homomorphism from \mathcal{M} into \mathcal{A} . \square

We can obtain an analogous result for UGMT in terms of complete OCAs. We define what it means for a lattice to be complete – a second-order property similar to the fusion operator in UGMT:

Definition 3.13 Let $\langle L, \cdot, +, 0, 1 \rangle$ be lattice. It is complete if and only if it is closed under arbitrary meets, that is

$$\forall S \subseteq L \exists x \in L : x = \prod S$$

A complete lattice is also complete under arbitrary joins, i.e.,

$$\forall S \subseteq L \exists x \in L : x = \sum S$$

These so-called fusion operators \sum and \prod are often alternatively denoted as \bigwedge and \bigvee , respectively. We call a contact algebra complete if its underlying lattice is complete.

Then, the following corollary immediately follows:

Corollary 3.2 Every model \mathcal{M} of UGMT is homomorphic to some complete OCA.

Proof With \mathcal{M} being a model of UGMT, it is also a model of UCMT which is homomorphic to some OCA. By (UGMT.1) and (UGMT.2) it is complete. \square

This enables us to focus on the topological representability or embeddability of (complete) OCAs in order to study representability of all the models of UCMT and of UGMT.

4 Mereotopologically representable complete OCAs

The study of topological representability of algebraic structures has a long tradition established by the seminal work by [43] on the duality between Boolean algebras and the topological spaces now known as Stone spaces. Since then, many generalizations thereof have been found. Here, we are not interested in full duality, but rather in embeddings of OCAs (with lattices as core) in a topological space in a way that preserves the mereotopological structure, i.e. gives point-set interpretations to all lattice elements so that parthood and contact also have point-set interpretations that reflect their intended spatial meaning. If an OCA has such a topological representation or embedding, we call it *spatially representable*. But instead of giving a complete definition of spatial representability, we only partially define it by giving a few necessary conditions that must hold in a spatially representable OCA. Every OCA that satisfies these conditions is called *mereotopologically representable* (MT-representable). Then we have for all OCAs

$$\text{spatially representable} \Rightarrow \text{MT-representable}$$

but not its converse, i.e.

$$\text{MT-representable} \not\Rightarrow \text{spatially representable}$$

Nevertheless, by showing that MT-representable complete OCAs are pseudocomplemented and satisfy the Stone identity we can conclude the same for spatially representable complete OCAs. Thus, MT-representability restricts the behaviour of complementation in the lattice structure of the algebraic counterparts resulting from models of **UGMT**. Translated into the realm of the logical theories, we essentially show that all models of **UGMTs** that have some spatial representation must have an algebraic structure whose lattice is a Stonian p-ortholattice. This defines a weakest class of equidimensional mereotopologies with unique closures under arbitrary sums, arbitrary joins, and under complementation.

For this section, we assume a basic familiarity with topological spaces. A few words on our notation: A topological space $\langle X, \tau \rangle$ is defined by its universe X and its topology τ , the set of all open subsets of X . Sets are denoted by capital letters to distinguish them from lattice elements. $h(a)$ denotes the set that a lattice element a is represented by. The interior, closure, and complement (with respect to X) of a set A are denoted by $\text{int}(A)$, $\text{cl}(A)$, and \bar{A} . Set intersection, union, and inclusion are denoted by \cap , \cup , and \subseteq . The following set-theoretic equivalences in topological spaces are used without further mentioning:

Lemma 4.1 Let $\langle X, \tau \rangle$ be a topological space. Then for all $A, B \subseteq X$:

$$\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B) \quad \text{and} \quad \text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B).$$

4.1 MT-representability For an OCA to be spatially representable, we require that a lattice homomorphism into a set of subsets of X of a topological space $\langle X, \tau \rangle$ exists. The lattice operations \cdot and $+$ correspond to operations \sqcap and \sqcup defined over the subsets of X . They may map to standard set intersection \cap and union \cup in the topological space, though this is not required. Notice that as a lattice homomorphism, h must preserve joins and meet, i.e. $h(x \cdot y) = h(x) \sqcap h(y)$ and $h(x + y) = h(x) \sqcup h(y)$. In particular, we must have $h(x) \subseteq h(y) \iff x \cdot y = y$; in other words, the lattice order \leq and thus the parthood order P is preserved as subset inclusion \subseteq in the representing topological space.

We are now ready to define MT-representability of an OCA:

Definition 4.1 Let $\mathcal{A} = (\langle L, +, \cdot, \perp, 0, 1 \rangle, \mathbf{C})$ be an OCA.

It is called MT-representable if there is some topological space $\langle X, \tau \rangle$ and an injective lattice homomorphism h from L into the structure $\langle \mathcal{T}, \sqcap, \sqcup \rangle$ where $T \subseteq X$ for each $T \in \mathcal{T}$ and the following hold for all $x, y \in L$:

1. $h(1) = X$ and $h(0) = \emptyset$;
2. $\text{int}(h(x)) \cap \text{int}(h(y)) \subseteq h(x \cdot y) = h(x) \sqcap h(y) \subseteq \text{cl}(h(x) \cap h(y))$;
3. for all $x \neq 0$, $\text{int}(h(x)) \neq h(0) = \emptyset$;
4. $\text{int}(h(x)) \cap \text{int}(h(y)) \neq \emptyset \Rightarrow x \mathbf{C} y$ for all $x \in L$;
5. $\text{cl}(h(x)) \cap \text{cl}(h(y)) = \emptyset \Rightarrow x - \mathbf{C} y$ for all $x \in L$.

Condition (1) ensures that the space is not larger than necessary, while condition (2) ensures that the set that represents the meet of two entities differs only in boundaries from the point-set intersection of their representing sets. More specifically, the representation of the meet of two entities is not smaller than the intersection of the interiors of their representations and not larger than the intersection of the closures of their representations. If condition (2) holds, its dual must also hold:

$$(2^*) \text{int}(h(x) \cup h(y)) \subseteq h(x + y) = h(x) \sqcup h(y) \subseteq \text{cl}(h(x)) \cup \text{cl}(h(y))$$

Condition (3) ensures that everything apart from the zero entity has a non-empty interior. The conditions (4) and (5) ensure that contact is adequately interpreted so that two entities whose representations share a point are indeed in contact while if the closures of their representations do not share a point, they are not in contact. Finally, we know $x \cdot x^\perp = 0$ and $x + x^\perp = 1$ and conclude that $h(x) \sqcap h(x^\perp) = \emptyset$ and $h(x) \sqcup h(x^\perp) = X$. Then we deduce from Def. 4.1(2) that:

$$(6) \text{int}(\overline{h(x)}) \subseteq h(x^\perp) \subseteq \text{cl}(\overline{h(x)})$$

Special version of MT-representability are representability by regular closed (or regular open) sets or by regular sets as for the Boolean Contact Algebras (BCAs) with $x - \mathbf{C} y \leftrightarrow x < y'$ or the Stonian p-ortholattices with $x - \mathbf{C} y \leftrightarrow x \leq y^\perp$. In other words, lattices representable by regular closed sets of a topological space, such as BCAs, satisfy all conditions of Def. 4.1. Key here is that conditions (2) and (4), (5) are satisfied if we use \cap as \sqcap and have $\text{cl}(x) = x$ for all $x \in L$; (2) then simplifies to $\text{int}(h(x)) \cap \text{int}(h(y)) \subseteq h(x) \cap h(y) \subseteq h(x) \cap h(y)$ which is trivially true, while (4) and (5) amount to $\text{cl}(h(x)) \cap \text{cl}(h(y)) \neq \emptyset \Leftrightarrow x \mathbf{C} y$ which is satisfied once we define $x - \mathbf{C} y \leftrightarrow x < y$ as in BCAs [cf. 16]. For the representation of Stonian p-ortholattices by regular sets, we can choose $x \sqcap y = x \cap y \cap \text{int}(\text{cl}(x \cap y))$ to satisfy condition (2) while the conditions (4) and (5) are satisfied if we define $h(x) \cap h(y) \neq \emptyset \Leftrightarrow x \mathbf{C} y$ [cf. 2]. Now we can prove the first property of MT-representable complete OCAs:

Theorem 4.2 A MT-representable complete OCA is pseudocomplemented.

Proof Suppose $\mathcal{A} = (\langle L, +, \cdot, \perp, 0, 1 \rangle, \mathbf{C})$ is a MT-representable complete OCA that is not pseudocomplemented. Then there must exist a lattice element $x \in L$ with two or more distinct but incomparable meet-complements $x_1^\perp, \dots, x_n^\perp \in L$. We have $x \cdot x_i^\perp = 0$ for $1 \leq i \leq n$ but for any i, k so that $1 \leq i, k \leq n$ and $i \neq k$ neither $x_i^\perp \leq x_k^\perp$ nor $x_k^\perp \leq x_i^\perp$. Then $h(x) \cap h(x_i^\perp) = \emptyset = h(x) \cap h(x_k^\perp)$ while neither $h(x_i^\perp) \subseteq h(x_k^\perp)$ nor $h(x_k^\perp) \subseteq h(x_i^\perp)$. By completeness of the lattice, there exists a unique sum $x^\perp \in L$ so that $x^\perp = \sum \{x_i^\perp \mid 1 \leq i \leq n\}$.

We will now show that $h(x) \cap h(x^\perp) = h(0)$ by contradiction. Suppose this is not the case, i.e., that $h(x) \cap h(x^\perp) \supsetneq h(0)$. Then there exists a set z that is a subset of both $h(x)$ and $h(x^\perp)$. By Def. 4.1(3) it has an interior $\text{int}(z) \neq 0$ that is also a subset of both $h(x)$ and $h(x^\perp)$. Because $h(x^\perp) \subseteq \sum\{\text{int}(h(x_i^\perp)) \mid 1 \leq i \leq n\}$, this open set $\text{int}(z)$ must be in one of the $\text{int}(h(x_i^\perp))$. Then $\text{int}(z) \subseteq h(x_i^\perp)$ for some i since $\text{int}(A) \subseteq A$ for all sets A . But then $\text{int}(z) \subseteq h(x \cdot x_i^\perp)$ for some i , a violation of our assumption that $x \cdot x_i^\perp = 0$. Hence, by $x^\perp \geq x_i^\perp$ for all i and $x \cdot x^\perp = h(0)$ the lattice element x^\perp is a pseudocomplement of x .

Furthermore, for an infinite ascending chain of meet-complements x_i there must exist by completeness a greatest one, namely its sum $\sum\{x_i\}$ which is by definition the pseudocomplement of $x \in L$. \square

The restriction to complete lattices essentially shifts the focus from **UCMT** to **UGMT**. Notice, however, that all discrete models of **UCMT** are trivially complete.

Now we prove that in a MT-representable OCA the Stone identity must also hold. First, recall that a pseudocomplemented ortholattice is also quasicomplemented, which also applies to contact algebras defined over those lattices. In the following, we utilize the fact that MT-representable complete OCAs are quasicomplemented to prove that they satisfy the Stone property. We exploit the fact that $h(x) \rightarrow h(x^{++})$ is an interior mapping in the topological sense for a quasicomplemented OCA given condition 2 of Def. 4.1 [cf. 30], i.e.,

$$h(x^{++}) = \text{int}(h(y)) \quad (+)$$

This is well known for Boolean lattices which are representable by the regular open sets of a topological space. More generally, it can be justified by considering that by the definition of a quasicomplement, x^+ is the smallest entity so that $x + x^+ = 1$. We have then $h(x^+ + x^{++}) = h(x^+ + x^{\perp+}) = h(x^+) \cup h(x^{\perp+}) = h(1) = X$ which is an open set in every topological space.

Analogously, $h(x) \rightarrow h(x^{**})$ is a closure mapping in the representation of a pseudocomplemented OCA given condition 2 of Def. 4.1, i.e.,

$$h(x^{**}) = \text{cl}(h(y)) \quad (*)$$

We further need the following result from [30]:

Lemma 4.3 Let $\langle L, +, \cdot, *, \perp, 0, 1 \rangle$ be a p-ortholattice. Then we have

1. $a^{**} = (a^{++})^{**}$
2. $a^{++} = (a^{**})^{++}$

We are now in the position to prove the Stone identity for MT-representable, quasicomplemented OCAs:

Theorem 4.4 A MT-representable OCA $\mathcal{A} = \langle \langle L, +, \cdot, \perp, +, 0, 1 \rangle, \mathbf{C} \rangle$ satisfies $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ for all $x, y \in L$.

Proof Suppose $\mathcal{A} = \langle \langle L, +, \cdot, \perp, +, 0, 1 \rangle, \mathbf{C} \rangle$ is a MT-representable quasicomplemented OCA. Let $x, y \in L$ denote two arbitrary lattice elements. We prove the two directions $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ and $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$ individually.

First $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ follows from

$$\begin{aligned}
h((x \cdot y)^{++}) &= \text{int}(h(x \cdot y)) && (+) \\
&\subseteq \text{int}(\text{cl}(h(x)) \cap \text{cl}(h(y))) && \text{Def. 4.1(2)} \\
&= \text{int}(h(x^{**}) \cap h(y^{**})) && (*) \\
&= \text{int}(\text{int}(h(x^{**}) \cap h(y^{**}))) && \text{int}(\text{int}(A)) = \text{int}(A) \\
&= \text{int}(\text{int}(h(x^{**})) \cap \text{int}(h(y^{**}))) && \text{Lemma 4.1} \\
&= \text{int}(h(x^{**++}) \cap h(y^{**++})) && (+) \\
&\subseteq h(x^{**++} \cdot y^{**++}) && \text{Def. 4.1(2)} \\
&= h(x^{++} \cdot y^{++}) && \text{Lemma 4.3}
\end{aligned}$$

For the other direction, $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$, suppose $(x \cdot y)^{++} \not\subseteq x^{++} \cdot y^{++}$. Then $h((x \cdot y)^{++}) \not\subseteq h(x^{++} \cdot y^{++})$ and there must exist a non-empty set z so that $h(z) \subseteq h(x^{++} \cdot y^{++})$ but $h(z) \not\subseteq h((x \cdot y)^{++})$. By Def. 4.1(3), we know that $\text{int}(h(z))$ is non-empty; hence we assume

$$\text{int}(h(z)) \subseteq \text{int}(h(x^{++} \cdot y^{++})) \quad (\text{assumption})$$

while $\text{int}(h(z)) \not\subseteq \text{int}(h((x \cdot y)^{++}))$ is contradicted by the following computation:

$$\begin{aligned}
\text{int}(h(z)) &\subseteq \text{int}(h(x^{++} \cdot y^{++})) && (\text{assumption}) \\
&\subseteq \text{int}(\text{cl}(h(x^{++}) \cap h(y^{++}))) && \text{Def. 4.1(2)} \\
&= \text{int}(\text{cl}(\text{int}(h(x)) \cap \text{int}(h(y)))) && (+) \\
&\subseteq \text{int}(\text{cl}(h(x \cdot y))) && \text{Def. 4.1(2)} \\
&= \text{int}(\text{int}(\text{cl}(h(x \cdot y)))) && \text{int}(\text{int}(A)) = \text{int}(A) \\
&= \text{int}(\text{int}(h((x \cdot y)^{**}))) && (*) \\
&= \text{int}(h((x \cdot y)^{**++})) && (+) \\
&= \text{int}(h((x \cdot y)^{++})) && \text{Lemma 4.3}
\end{aligned}$$

With h being an injective lattice homomorphism, we conclude $(x \cdot y)^{++} = x^{++} \cdot y^{++}$. \square

This leads us to the definition of SPOCAs as a subclass of OCAs which contains all complete OCAs that are MT-representable. SPOCAs can be axiomatized algebraically by the theory $T_{\text{SPOCA}} \cup \{L2^{\vee}-L6^{\vee}, L2^{\wedge}-L4^{\wedge}, O1', O2', O3', PC1, PC2', \text{ and } PC2'', S, C0-C3\}$, see Appendix A for the axioms and see [48] for more explanations and a reduction of this non-minimal theory.

Definition 4.2 A Stonian pseudocomplemented and orthocomplemented contact algebra (SPOCA) is a structure $(\langle L, \cdot, +, \perp, \overset{\perp}{+}, 0, 1 \rangle, \mathbf{C})$ so that

1. $\langle L; \cdot, +, \perp, 0, 1 \rangle$ is an ortholattice;
2. $\langle L, \cdot, +, \overset{\perp}{+}, 0, 1 \rangle$ is a quasicomplemented lattice;
3. $(a + b)^{\perp} = a^{\perp} \cdot b^{\perp}$ for all $a, b \in L$;
4. \mathbf{C} satisfies (C0) to (C3).

The following corollary summarizes our result of this section:

Corollary 4.5 A MT-representable complete OCA is a complete SPOCA.

As a consequence, from now on we can focus our attention to SPOCAs without worrying that other spatially representable classes of contact algebras may be overlooked. The only case we have not accounted for are strictly non-complete lattices. It is however unlikely that any such class is of relevance for a spatially representable mereotopology.

5 Closure operations in SPOCAs

In this section, we give a mereological and a topological definition of each of the closure operations sum, intersection, complement, and universal; closely adhering to the definitions presented in [7]. We investigate whether each of the four closure operations are defined in either (or in both) ways in general SPOCAs. For those mereological or topological closure operations that are not entailed, we identify equivalent algebraic properties. Surprisingly, very few such additional properties are necessary; the necessary ones primarily arise from complements being defined mereologically or topologically. If we define complements mereologically, the arising SPOCAs are distributive, while defining complements topologically allows SPOCAs whose underlying Stonian p-ortholattices are non-distributive. In the later case, the contact relation must be more restricted. The resulting two main types of SPOCAs are explored in detail in Section 6.

Generally we expect each of the closure operations to be defined at least mereologically or topologically. But from an ontologically sound theory of mereotopology, we expect further that all closure operations are defined consistently, e.g. either all are defined mereologically or all are defined topologically. We use the following terminology, the axioms follow shortly:

Definition 5.1 A UCMT is *M-closed* iff it satisfies $M-I_{UCMT}$, $M-S_{UCMT}$, and $M-C_{UCMT}$.

Definition 5.2 A UCMT is *T-closed* iff it satisfies $T-I_{UCMT}$, $T-S_{UCMT}$, and $T-C_{UCMT}$.

Definition 5.3 A UCMT is *T'-closed* iff it satisfies $T-I_{UCMT}$, $T-S_{UCMT}$, and $T-C'_{UCMT}$.

A UCMT is then *coherently closed* (C-closed) if it is defined in one of those three ways:

Definition 5.4 A UCMT is *C-closed* iff it is M-closed, T-closed, or T'-closed.

Ideally, the closure operations can be defined mereologically and topologically at the same time. Then we call it *MT-closed*.

Definition 5.5 A UCMT is *MT-closed* iff it is

1. M-closed, and
2. T-closed or T'-closed.

We use all these properties both for the logical theories as well as for their corresponding algebraic theories.

All lemmas throughout this and the subsequent section have been proved using the automated theorem prover Prover9 [34] unless otherwise stated. Most proofs are omitted since they contribute only little insight; proof inputs and outputs can be found at www.cs.toronto.edu/~torsten/CA/.

5.1 Mereological closure operations The closure operations intersection, sum, and complementation can be defined mereologically as following. It is easily verified that these are consistent with (UCMT.6) and (UCMT.7).

$$\begin{aligned} (\text{M-I}_{UCMT}) \quad & \forall w [P(w, x \odot y) \leftrightarrow (P(w, x) \wedge P(w, y))] && \text{(Intersection)} \\ (\text{M-S}_{UCMT}) \quad & \forall w [O(w, x \oplus y) \leftrightarrow (O(w, x) \vee O(w, y))] && \text{(Sum)} \\ (\text{M-C}_{UCMT}) \quad & \forall w [O(w, \ominus x) \leftrightarrow \neg P(w, x)] && \text{(Complement)} \end{aligned}$$

In the sequel we will exclusively use the algebraic equivalents of these axioms as found in Appendix A. These differ only slightly from the above axioms to account for the additional bottom element 0 in a contact algebra, see Lemma B.1 in Appendix B for the proof of the equivalence of the two versions. In the algebraic versions of the axioms \wedge and \vee denote meet and joint, while the logical connectives are written as $\&$ and $|$.

Notice that the universal u is always defined mereologically as $\forall x P(x, u)$. Moreover, we can easily prove that the algebraic equivalents of (M-I_{UCMT}) and (M-S_{UCMT}), i.e. (M-I) and (M-S), are theorems in SPOCAs.

Lemma 5.1 $T_{SPOCA} \models \text{M-I}$

Lemma 5.2 $T_{SPOCA} \models \text{M-S}$

(M-C) does not necessarily hold in SPOCAs. Defining complementation mereologically requires the SPOCA to be uniquely complemented and thus distributive and Boolean. The necessary axiom (Uni) postulating unique complementation can be found in Appendix A.

Lemma 5.3 $T_{SPOCA} \models \text{M-C} \leftrightarrow \text{Uni}$

Proof Since unicomplemented ortholattices are Boolean and vice versa it suffices to show that a unicomplemented SPOCA satisfies the algebraic equivalence of (M-C): $z \cdot x^\perp \neq 0 \leftrightarrow z \not\leq x$ and that a SPOCA satisfying this property is unicomplemented. This has been done using the automated theorem prover. \square

For the sums and complements to be unique, we further need extensionality of O postulated as following. Recall that $\neg O(x, y) \iff x \wedge y = 0$.

$$(\text{O-Ext}) \quad \forall z (z \wedge x = 0 \leftrightarrow z \wedge y = 0) \leftrightarrow x = y \quad \text{(O-extensionality)}$$

But from (M-C) we can already prove extensionality of O:

Lemma 5.4 $T_{SPOCA} \cup \text{M-C} \models \text{O-Ext}$

We obtain the following corollary on the effects of mereological closures in SPOCAs:

Corollary 5.5 A SPOCA is M-closed iff it is unicomplemented. A M-closed SPOCA is O-extensional.

5.2 Topological closure operations The closure operations intersection, sum, and complementation can be defined topologically as following. Again, their algebraic versions are found in Appendix A with Lemma B.2 in Appendix B proving the equivalence of both versions. It is easily verified that these are consistent with (UCMT.6) and (UCMT.7). There are two slightly distinct ways of defining topological complements, denoted by (T-C) and (T-C').

$$(\text{T-I}_{UCMT}) \quad \forall w [C(w, x \odot y) \rightarrow (C(w, x) \wedge C(w, y))] \quad \text{(Intersection)}$$

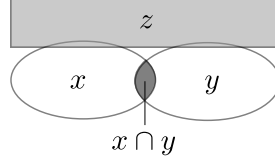


Figure 5 Two regions x and y connected to z whose set intersection is not connected to z due to the non-transitive nature of contact.

$$(T-S_{UCMT}) \quad \forall w[C(w, x \oplus y) \leftrightarrow (C(w, x) \vee C(w, y))] \quad (\text{Sum})$$

$$(T-C_{UCMT}) \quad \forall w[P(w, \ominus x) \leftrightarrow \neg C(w, x)] \quad (\text{Complement})$$

$$(T-C'_{UCMT}) \quad \forall w[PP(w, \ominus x) \leftrightarrow \neg C(w, x)] \quad (\text{Alternative complement})$$

Notice that since the universal is always defined mereologically as $\forall x P(x, u)$, it is also automatically defined topologically as $\forall x C(x, u)$. However, this does not guarantee the topological uniqueness of the universal, i.e. that $\forall y [(\forall x C(x, y)) \rightarrow y = u]$ holds.

Intersections are always defined topologically in SPOCAs. Notice however that (T-I) only contains a simple implication and not a biconditional. The reverse direction is not desirable as Fig. 5 illustrates.

Lemma 5.6 $T_{SPOCA} \models \text{T-I}$

Proof Follows directly from (C3). \square

Moreover, SPOCAs satisfy one direction of the implication in the axiom (T-S).

Lemma 5.7 $T_{SPOCA} \models \forall x[xC(y+z) \leftarrow (xCy \mid xCz)]$

Proof Follows directly from (C3). \square

Since the reverse direction of (T-S) does not always hold, we can use (C4) to guarantee that sums are defined topologically in SPOCAs, i.e. if an element x is connected to another element z , it is also connected to one of the parts of z that make up z .

$$(C4) \quad xC(y+z) \rightarrow xCy \mid xCz \quad (\text{Topological sum})$$

Lemma 5.8 $T_{SPOCA} \cup C4 \models \text{T-S}$

Proof Immediately follows from Lemma 5.7. \square

5.2.1 Topological complement operation Now we turn to the complement. We have two options, using either (T-C_{UCMT}) or (T-C'_{UCMT}). We first study (T-C_{UCMT}) and then proceed with (T-C'_{UCMT}). In SPOCAs, (T-C_{UCMT}) is captured algebraically by (C5) which requires an element to be in contact to all elements that are not parts of its orthocomplement. In particular, for any x , $x - Cx^\perp$.

$$(C5) \quad zCx \leftrightarrow z \not\leq x^\perp \quad (\text{Topological complement})$$

Interestingly, (C5) alone is sufficient to ensure that (T-S) holds and that C is extensional, i.e. (C4) and (C-Ext) are satisfied in SPOCAs which satisfy (C5). (C-Ext) expresses extensionality of C, that is, two elements are considered identical if they are in contact to exactly the same elements. C-extensionality is equivalent to requiring that a mereotopology can be reconstructed from contact as the only primitive relation. It further ensures topological uniqueness of the universal.

$$(C\text{-Ext}) \quad \forall z(z\mathbf{C}x \leftrightarrow z\mathbf{C}y) \leftrightarrow x = y \quad (C\text{-extensionality})$$

Lemma 5.9 $T_{SPOCA} \cup C5 \models C4$

Lemma 5.10 $T_{SPOCA} \cup C5 \models C\text{-Ext}$

Moreover, (Int) must hold in SPOCAs satisfying (C5). This seems, however, coincidental and owed to the fact that elements are disconnected from their complements, that is, (\neg Con) holds. Despite its name, (\neg Con) is not the negation of (Con) but the exact opposite assumption. (Con) is inconsistent with a non-trivial SPOCA satisfying (C5). (Int) and (Con) have previously only been used in the context of contact algebras with Boolean lattices, but easily generalize to SPOCAs. In our study we include (Int) only for completeness purposes, it is not motivated by or directly related to the closure operations.

$$\begin{array}{ll} (\text{Con}) & \forall x \neq 0, 1[x\mathbf{C}x^\perp] \quad (\text{Connected complements}) \\ (\neg\text{Con}) & \forall x[x-\mathbf{C}x^\perp] \quad (\text{Disconnected complements}) \\ (\text{Int}) & \forall x, y[x-\mathbf{C}y \rightarrow \exists z(x-\mathbf{C}z \ \& \ y-\mathbf{C}z^\perp)] \quad (\text{Interpolation}) \end{array}$$

Lemma 5.11 $T_{SPOCA} \cup C5 \models \neg\text{Con}$

Proof Choose $y = x^\perp$ in (C5) to obtain $\neg x\mathbf{C}x^\perp$. \square

Lemma 5.12 $T_{SPOCA} \cup C5 \models \text{Int}$

Proof We show that choosing $z = x^\perp$ in (Int) always evaluates to true. We obtain $x-\mathbf{C}y \rightarrow (x-\mathbf{C}x^\perp \wedge y-\mathbf{C}x^{\perp\perp})$. By Lemma 5.11 it is sufficient to prove $\forall x, y[x-\mathbf{C}y \rightarrow y-\mathbf{C}x^{\perp\perp}]$ which is with $x = x^{\perp\perp}$ the trivially true inverse of (C2). \square

We obtain the following corollaries on the effect of topological closures in SPOCAs:

Corollary 5.13 A SPOCA is T-closed iff it satisfies (C5). A T-closed SPOCA is C-extensional and satisfies (C4), (\neg Con), and (Int).

Finally, we verify that (C5) and (Uni) are independent of another, i.e. that there exists SPOCAs that satisfy (C5) but are not uniquely complemented and that there exist SPOCAs with a Boolean lattice that do not satisfy (C5). Both results are not very surprising.

Lemma 5.14 $T_{SPOCA} \cup \text{Uni} \not\models C5$

Lemma 5.15 $T_{SPOCA} \cup C5 \not\models \text{Uni}$

5.2.2 Quasi-topological complement operation Now we turn to ($\text{T-C}'_{UCMT}$) as an alternative to the axiom (T-C_{UCMT}) for defining complements topologically. ($\text{T-C}'_{UCMT}$) is captured algebraically by (C5').

$$(C5') \quad x \neq 0 \mid z \neq 1 \mid (x \neq 1 \ \& \ z \neq 0) \rightarrow [z\mathbf{C}x \leftrightarrow z \not\prec x^\perp] \quad (\text{Alternative topological complement})$$

Obviously, (C5) and (C5') are mutually inconsistent but what are the consequences of using (C5') instead of (C5) to define complements? (C5') requires an element to be connected to all other elements which are not proper parts of its (ortho-)complement, i.e. (Con) is a theorem.

Lemma 5.16 $T_{SPOCA} \cup C5' \models \text{Con}$

By Lemma 5.16 (C5') is not really a topological definition of complementation since complements are connected, i.e. $x\mathbf{C}x^\perp$. Truly topological complements are complementary with respect to C. In a SPOCA that satisfies (C5'), (C-Ext) does not necessarily hold. In particular, we can have $\exists x, y [x \neq y \ \& \ \forall z (x\mathbf{C}z \ \& \ y\mathbf{C}z)]$. Then, the universal is no longer topologically unique; this requires (Dis) in addition. For that reason, we refer to (C5') as a *quasi-topological complement*.

In the presence of (C5'), (Int) is also theorem of SPOCAs.

Lemma 5.17 $T_{SPOCA} \cup C5' \models Int$

Most important here is that (C5') forces SPOCAs to be uniquely complemented and thus Boolean. Then, mereological complements are also guaranteed to exist and (O-Ext) directly follows.

Lemma 5.18 $T_{SPOCA} \cup C5' \models Uni$

On the reverse, unicomplementation is not sufficient to ensure that (C5') is satisfied.

Lemma 5.19 $T_{SPOCA} \cup Uni \not\models C5'$

Therefore the class of SPOCAs satisfying (C5') is an extensions of the SPOCAs that have a Boolean lattice structure. Those that additionally satisfy (C4) have all closure operations defined mereologically and topologically except for the complement which is defined mereologically but only quasi-topologically. We obtain three more corollaries on the effect of topological closures in SPOCAs:

Corollary 5.20 A SPOCA is T'-closed iff it satisfies (C4) and (C5').

Corollary 5.21 A T'-closed SPOCA is unicomplemented, O-extensional, and satisfies (Con) and (Int).

Corollary 5.22 A T'closed SPOCA is M-closed.

6 Coherently closed MT-representable UCMTs

We already mentioned that a UCMT is only ontologically coherent if it is M-closed, T-closed, or T'-closed. Corollary 5.22 lets us now simplify the definition of C-closure for MT-representable UCMTs, since we established in Section 4 that MT-representable UCMTs result in SPOCAs as their algebraic counterparts.

Corollary 6.1 A MT-representable UCMT is C-closed iff it is T-closed or M-closed.

Now we can use the Corollaries 5.5 and 5.13 to identify the two weakest theories of C-closed MT-representable UCMTs and explore the theories with stronger topological or mereological closure conditions. A particular emphasis will be on theories that admit discrete models, i.e. theories allowing models that contain atomic entities.

6.1 M-closed MT-representable UCMTs Because M-closed SPOCAs are unicomplemented, they must have a Boolean lattice:

Corollary 6.2 The algebraic counterpart of a M-closed UCMT has a Boolean lattice.

Proof Follows from unicomplemented ortholattices being Boolean [5]. \square

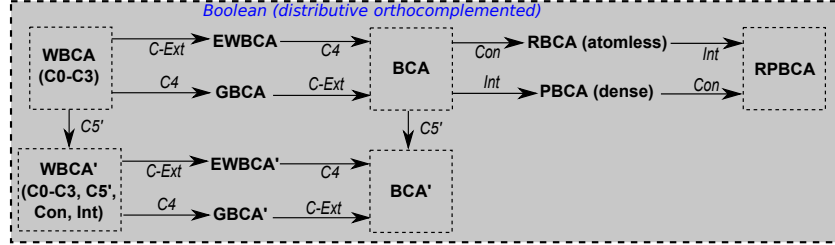


Figure 6 The classes of M-closed contact algebras and their extension relations amongst them indicated by arrows. For each class more than a single contact relation may exist. For example, on BCAs contact defined as overlap $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$ or as standard contact $x-\mathbf{C}y \leftrightarrow x < y'$ are two distinct extensional contact relations. On the other side, there are Boolean algebras that only allow strictly non-weak and/or extensional contact relations.

Many of the contact algebras previously studied in the literature have Boolean lattices and satisfy (C0)–(C3) [see 15; 32; 39]. The most important ones are:

Definition 6.1 A contact algebra $(\mathcal{L}, \mathbf{C})$ in which \mathcal{L} is a Boolean lattice is a

1. Generalized Boolean contact algebra (GBCA) if \mathbf{C} satisfies (C4);
2. Boolean contact algebra (BCA) if \mathbf{C} satisfies (C4) and (C-Ext);
3. RCC algebra (RBCA) if \mathbf{C} satisfies (C4), (C-Ext), and (Con);
4. Proximity BCA (PBCA) if \mathbf{C} satisfies (C4), (C-Ext), and (Int).

For a more comprehensive overview of the different classes of contact algebras and their relationships to each other we refer to [28]. Contact algebras that have Boolean lattices but do not satisfy (C4) are even weaker than GBCAs; we call them *weak Boolean contact algebras* (WBCA):

Definition 6.2 A weak Boolean contact algebra (WBCA) is a contact algebra $(\mathcal{L}, \mathbf{C})$ in which \mathcal{L} is a Boolean lattice \mathcal{L} .

As illustrated by the model in Fig. 7(a), there do exist WBCAs that satisfy neither (C4) nor (C-Ext). Thus, the class WBCAs is strictly more general than both EWBCAs (to be introduced shortly) and GBCAs. WBCAs are the weakest algebraic structures resulting from an MT-representable **UCMT** that is M-closed. WBCAs admit atoms and in particular finite models as Fig. 7(a) shows.

Theorem 6.3 A M-closed MT-representable **UCMT** has an algebraic structure $(\mathcal{L}, \mathbf{C})$ whose lattice \mathcal{L} is Boolean and whose contact relation satisfies (C0) to (C3).

This is a more general perspective of the results from [18] in which the different contact relations definable on Boolean algebras have been studied. The weakest contact relation in [18] already satisfies (C4), while Fig. 7(a) shows that there are weaker contact relations definable on a contact algebra with a Boolean lattice which may arise from M-closed **UCMTs** whose sums are not topologically closed, i.e., which violate (C4). We do not argue for the usefulness of these structures; in practice (C4) seems like a reasonable assumption. We only explore weaker M-closed contact algebras by showing what other contact relations are theoretically definable on a Boolean lattice.

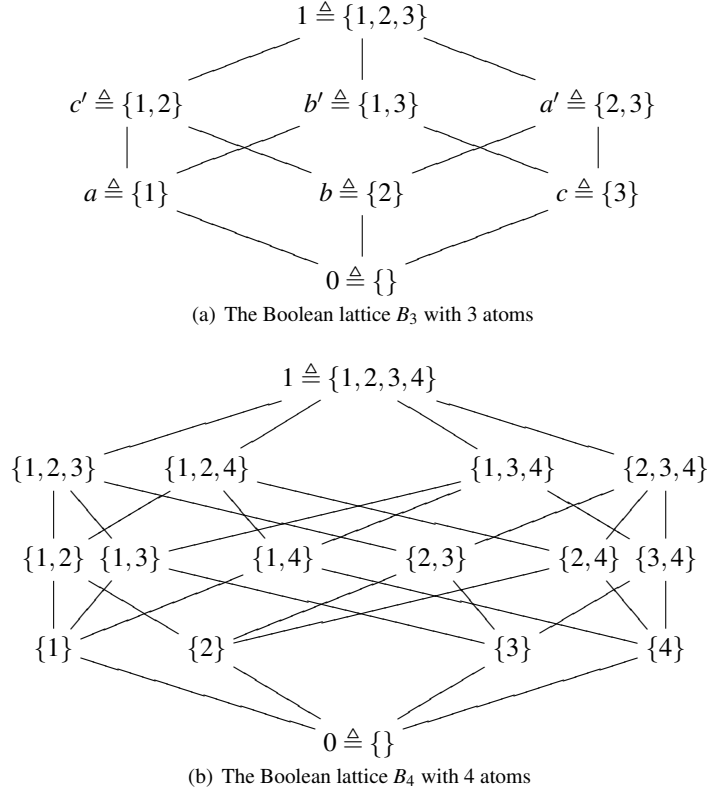


Figure 7 (a) B_3 with $\{(0,x)|x \in L\} \cup \{(a,b), (a,c), (b,c)\} \notin \mathbf{C}$ (and symmetric tuples) defining disconnection is a WBCA which does not satisfy (C-Ext) nor (C4). B_3 with $\{(0,x)|x \in L\} \cup \{(a,c)\} \notin \mathbf{C}$ (and symmetric tuples) is a non-extensional GBCA. The elements a', b', c' , and 1 are indistinguishable with respect to the contact relation.

(b) B_4 with $x < y' \rightarrow x - \mathbf{C}y$ defining disconnection except for $\{\{1,2\}, \{3,4\}\} \in \mathbf{C}$ results in an EWBCA not satisfying (C4).

WBCAs can be extended by (C-Ext) to obtain *Extensional Weak Boolean Contact Algebras* (EWBCA) or by (C4) to obtain the already defined GBCAs. EWBCAs are axiomatizable by the theory $T_{EWBCA} = T_{SPOCA} \cup \{Uni, C-Ext\}$, see Appendix A for the axioms.

Definition 6.3 An extensional weak Boolean contact algebra (EWBCA) is a WBCA $(\mathcal{L}, \mathbf{C})$ in which the contact relation \mathbf{C} satisfies (C-Ext).

Again, there exist EWBCAs whose contact relation do not satisfy (C4) (compare Fig. 7a). Unfortunately, in the following we show that in all EWBCAs not satisfying (C4), $x \mathbf{C} x'$ holds for some elements, while for atoms it cannot hold. In other words, the theory of EWBCAs extended by the negation of (C4) and (Atom) is inconsistent

with either of $(\neg\text{Con})$ and (Con) . For the proof we rely on the following result from [17] stating that (Dis) implies (C-Ext) in contact algebras and thus in WBCAs:

$$(\text{Dis}) \quad \forall x[x \neq 1 \rightarrow \exists y(y \neq 0 \wedge x - \mathbf{C}y)] \quad (\text{Disconnection})$$

Lemma 6.4 [17] $T_{WBCA} \models \text{C-Ext} \rightarrow \text{Dis}$

Lemma 6.5 $T_{EWBCA} \cup \neg\text{C4} \cup \neg\text{Con} \models \perp$

Proof We give an automatic proof showing that $T_{SPOCA} \cup \{\text{Uni}, \text{Dis}\} \cup \neg\text{C4} \cup \neg\text{Con} \models \perp$ for all non-trivial models. Since by Lemma 6.4 $T_{SPOCA} \cup \{\text{Uni}, \text{Dis}\}$ is strictly weaker than T_{EWBCA} , $(\neg\text{Con})$ is inconsistent with any non-trivial EWBCA. \square

EWBCAs that contain some atom are inconsistent with (Con) because the atom must be connected to its complement:

Lemma 6.6 $T_{EWBCA} \cup \text{Atom} \cup \text{Con} \models \perp$

Proof By Lemma 6.4 it is sufficient to prove that $T_{SPOCA} \cup \{\text{Uni}, \text{Dis}\} \cup \text{Atom} \cup \text{Con} \models \perp$. Let a be an atom of L . Then a' is a dual atom, i.e. only $1 > a'$. By overlap, a' is in contact to all elements except for a . Now by (Con) $a\mathbf{C}a'$, then $\forall y[a'\mathbf{C}y \leftrightarrow 1\mathbf{C}y]$ but $1 \neq a'$, a violation of (Dis) . This does not hold for a trivial model where $1' = 0$. \square

Therefore, all models of EWBCAs which do not satisfy (C4) but contain an atom suffer from this non-uniform interpretation of the contact relation — in particular all atomic, all atomistic, and all finite models of EWBCAs and, more generally, of WBCAs with (Dis) . That $x\mathbf{C}x'$ for atoms x is inconsistent with extensionality has been observed for BCAs in [38]. Our proofs are slightly stronger and show that this problem persists in the weaker theory WBCAs extended by (Dis) requiring a topologically unique universal element. The failure of $x\mathbf{C}x'$ for some elements is not by itself a concern; in a disconnected model one element may be isolated from the remaining space. However, the failure of $x\mathbf{C}x'$ for *all* atoms is a serious issue hinting to a weakness in the theory. Although it can be overcome by enforcing (C4) , this only creates other problems since (C4) and (C-Ext) together disallow any discrete models unless contact is reduced to overlap (which in turn reduces the theory to a pure mereology). The problem does not persist in WBCAs, for those we can prove that (Con) is consistent:

Lemma 6.7 $T_{WBCA} \cup \neg\text{C4} \cup \text{Atom} \cup \text{Con} \not\models \perp$

Proof Fig. 7(a) provides a counterexample. \square

What extensions of WBCAs are obtained if some of the closure operations are also defined topologically? Intersections are already defined topologically in WBCAs. If sums are defined topologically, we require (C4) and obtain GBCAs. If we define complements topologically by (C5) , we obtain PBCAs. Its discrete models again reduce contact to overlap. Finally, if we require neither sums nor complements to be defined topologically, but instead enforce C-extensionality, we obtain BCAs whose discrete models have overlap as the only feasible contact relation. Hence, amongst the different strengths of closure operations, the two classes WBCAs and GBCAs are the only algebraic theories of M-closed MT-representable UCMTs that admit non-atomless models with a contact relation different from overlap.

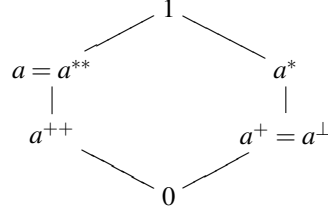


Figure 8 Let $\forall x, y[x \neq 0 \ \& \ x \leq y \rightarrow x\mathbf{C}y]$ and $a\mathbf{C}a^*$ (and symmetric tuples) define contact. Then the displayed lattice \mathcal{C}_6 together with \mathbf{C} defines a SPOCA $(\mathcal{C}_6, \mathbf{C})$ that satisfies (C5) and is thereby T-closed.

The extensions of WBCAs with the quasi-topological complements requires adding C5', resulting in MT-representable contact algebras that parallel those without C5', see Fig. 6. Amongst them, even the weakest theory WBCA' does not admit models with atoms:

Lemma 6.8 $T_{SPOCA} \cup C5' \cup \text{Atom} \models \perp$

6.2 T-closed MT-representable UCMTs T-closed MT-representable UCMTs have SPOCAs as algebraic counterparts which may be non-distributive as long as (C5) is satisfied:

Theorem 6.9 A T-closed MT-representable UCMT has an algebraic structure $(\mathcal{L}, \mathbf{C})$ in which \mathcal{L} is a Stonian p-ortholattice and \mathbf{C} satisfies (C0)–(C3) and (C5).

This non-distributive class of SPOCAs has been studied in depth in [30]; it is the algebraic equivalent of the subtheory RT^- of the mereotopology of [2].

Because such T-closed SPOCAs also satisfy (C4), intersections and sums are implicitly defined mereologically as well. The only real extension in terms of additional mereological closure operations requires complements to be mereologically defined, which in turn by Lemma 5.3 makes the lattice Boolean and thus results in a PBCA: (C-Ext) as well as (Int) are already entailed in all T-closed SPOCAs. This also means (C-Ext) extends T-closed SPOCAs nonconservatively, while (Con) is altogether inconsistent with T-closed SPOCAs. We already know that PBCAs are always atomless, hence the theory $SPOCA \cup (C5)$ is – amongst all possible extensions of T-closed MT-representable UCMTs by additional mereological closure operations – the only theory that admits atoms. Fig. 8 gives such a model.

6.3 MT-closed MT-representable UCMTs Sections 6.1 and 6.2 let us conclusively answer the question whether MT-closed MT-representable UCMTs exist and what their structure is. Such structures must be either M-closed and T-closed or only T'-closed, the latter implying that there are M-closed by Corollary 5.22. For the first case, the intersection of the respective minimal theories, i.e. of WBCAs and SPOCAs satisfying (C5), results in PBCAs that satisfy (C5). In these structures $(\neg\text{Con})$ is entailed; it requires that the contact relation is defined as overlap $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$ which reduces the theory to a pure mereology. For the second case, we get the BCAs with contact defined as (C5') as minimal theory. Hence, we have the following result:

Theorem 6.10 Every MT-closed MT-representable **UCMT** with $C \not\cong O$ has an algebraic structure that is a BCA $(\mathcal{L}, \mathbf{C})$ in which \mathbf{C} satisfies $(C5')$.

We also have negative results on the existence of MT-representable **UCMTs** with $C \not\cong O$:

Theorem 6.11 No M-closed and T-closed MT-representable **UCMT** with $C \not\cong O$ exists.

Theorem 6.12 No MT-closed MT-representable **UCMT** with $C \not\cong O$ and with discrete models exists.

7 Summary and discussion

Our exploration revealed two weakest classes of potentially spatially representable complete OCAs that correspond to extensions of **UCMT**. On the one hand, WBCAs are the weakest class in which all closure operations are defined mereologically. On the other hand, SPOCAs with $(C5)$ are the weakest class in which all closure operations are defined topologically. A third candidate class, SPOCAs with $(C5')$ turned out to further extend WBCAs and thus not a weakest class in its own right. We are not aware of full embedding theorems for these two weakest classes of contact algebras. This remains to be investigated in the future.

7.1 Spatially representable contact algebras with discrete models Amongst the spatially representable OCAs, the classes allowing discrete models are of particular interest. Although space is potentially infinitely divisible according to Aristotle, in practical applications any concrete model of space will have ‘atoms’ at some level, i.e. there is some finest granularity. This granularity is usually determined by the precision of available data or measurement devices (think of satellite images vs. microscopic pictures) or the precision we want to reason at (think of a car navigation system vs. the accurate description of surface chemistry). For a generic Ontology (in the philosophical sense) of space discrete models might not be that important, but for any specific domain we want to be able to specify models completely e.g. by explicitly listing a finite set of regions and the primitive relations (such as connection and parthood) amongst them. Such a specification should be consistent with the theory and not a mere approximation thereof. Many mereotopologies, e.g. the RCC (corresponding to RBCAs), prevent the existence of atomic regions by including a divisibility axioms, i.e. requiring the existence of an interior part for each region. Such theories do not allow us to list all atomic regions of a specific model. Of course, approximations of such models are possible, but these approximations have different model-theoretic properties. This has an important consequence: the construction of and the reasoning with specific models using a theory consistent with discrete, and especially finite, models can be achieved using standard theorem provers, which is not possible for mereotopological theories that only admit infinite models.

Which extensions of the two weakest classes of MT-representable OCAs allow discrete models, i.e., are not atomless? We showed that non-atomless WBCAs and EWBCAs have contact relations that behave erratic with regard to contact amongst complements. While the stronger BCAs and extensions thereof do not suffer from this problem, their discrete models are only of mereological nature, i.e. $C \cong O$ [16]. Similarly, WBCAs satisfying $(C5')$ rule out discrete models by Lemma 6.8. This

leaves GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ as the only (amongst all combinations of mereological and topological closure operations) MT-representable OCAs that admit discrete models. These two classes can be characterized as following:

1. GBCAs in which all closure operations are defined mereologically while sums and intersection are also defined topologically. In general, GBCAs are consistent with either of (Con) or (\neg Con). The entities in such algebraic structures are representable by either (1) only regular open, (2) only regular closed, or (3) unrestricted point sets (with point-set intersections, unions, and complements). In the second case (Con) must hold while in the other cases (\neg Con) must hold. The lattices underlying this class are distributive, i.e., parthood is distributive with respect to sum and intersections.
2. The subclass of SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ as weakest contact algebras defining all closure operations topologically while sums and intersections are also defined mereologically. Due to the topological nature of complements, (\neg Con) must hold. The representation of such algebraic structures must include both regular open and regular closed sets, since each regular closed set has a regular open set as complement and vice versa. In this class, the underlying lattices – and thus the parthood relation – may be non-distributive.

Indeed, GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ exemplify the two ways of constructing discrete mereotopologies discussed in [33]. SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ constitute a C-extensional theory with classical topological operators in which each entity, in particular each atom, is ‘duplicated’ as open and as closed set, while GBCAs define an O-extensional theory without classical topological operations, i.e., that do not distinguish regions with identical closures.

7.2 Spatially representable Whiteheadian mereotopology In [47], Whitehead originally proposed a C-extensional mereotopology and defined atoms as regions without proper parts. He thereby implied the existence of discrete models. Unfortunately, as Theorem 6.12 shows, no MT-closed MT-representable mereotopology with discrete models can exist. In fact, the only theory that (1) allows atoms, (2) is C-extensional, and (3) is MT-representable are the SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ defining contact – assuming that this class of SPOCAs can be further strengthened to a class of spatially representable SPOCAs; cf. [48] for work in this direction. From [30] we know that such theory is also definable by a single mereological primitive P (the partial order relation \leq in the lattices) or by a single topological primitive C ; it seems to seamlessly bridge the gap between mereology and topology. But at the same time, Whitehead never distinguished sets with identical closures. We can understand this as an implicit condition for representations by closed regions (or, dually, by only open regions); in fact many researchers followed this understanding of Whitehead’s intentions. He entices us to believe that the two assumptions, namely existence of atoms and representability by closed regions, are consistent. However, SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ as the only remaining candidate for true Whiteheadian mereotopology do rely on this difference between interiors and closures. If the distinction between interiors and closures is removed, these models collapse into Boolean contact algebras, compare [49], and thereby prevent a meaningful definition of contact apart from overlap in discrete models. With this stricter requirement of representability by only closed sets, no discrete region-based theory in the intention of Whitehead is definable [see also 20; 35]. Further research on theories of qualitative discrete space must

therefore concentrate on non-topological, such as graph-based, approaches. Independently, mereotopologies accommodating regions of various dimensions deserve more attention.

There are others ways out of this dilemma as demonstrated in the literature. If we do not insist on discrete models, RBCAs and the equivalent logical theory RCC, provide a truly Whiteheadian account of *continuous* space. One spatial representation thereof is the complemented disk algebra consisting of all simple closed regions of, e.g., \mathbb{R}^2 as described in detail in [31]. If we abandon C-extensionality instead we can rely on GBCAs. Non-extensional theories have also been used for defining multi-dimensional mereotopologies [22; 38]. The rationale for giving up C-extensionality is simple [38]: C-extensionality is a principle that holds in the perfect world where we can always find smaller parts that distinguish two distinct entities. If finite models are considered as models with limited accuracy, i.e. as approximations of continuous models, C-extensionality may be violated because the distinction in the contact between two entities may be too small a part so that it is lost in the approximation.

An alternative parsimonious way out of this dilemma is to abandon $\forall xC(x, \ominus x)$ instead. The non-distributive SPOCAs with $xCy \leftrightarrow x \not\leq y^\perp$ allow such choice. At first sight it seems to be a surprising choice since well-behaviour of lattices is usually associated with distributivity. But as we have shown in [30], the non-distributive lattices in question (Stonian p-ortholattices and restrictions) behave nicely even without distributivity. In particular, these structures also satisfy the DeMorgan laws and stop only short of being Boolean. We thereby are able to answer the question posed in [13] asking what kind of structures should be considered the standard model of a non-distributive contact algebra for the case of spatially representable contact algebras. The standard (and only) models of spatially representable complete non-distributive contact algebras are the regular sets of a topological space.

Notice that there is no need to completely abandon (Con). If we define an additional ‘attachment’ relation A from C as $A(x, y) \leftrightarrow [C(x, y^{**}) \vee C(x^{**}, y)] \wedge \neg C(x, y)$, we can prove $\forall xA(x, \ominus x)$ in a connected space even if $\forall x\neg C(x, \ominus x)$. Attachment is a stronger relation than contact defined in SPOCAs as $\neg C(x, y) \leftrightarrow x \leq y^\perp$, but weaker than *weak contact* $WCont$ as defined in [2]. Moreover, C and A make the distinction between the intended interpretations of ‘sharing a point’ and ‘overlapping neighborhoods’ clear.

7.3 Conclusion This work treated mereotopology with unique closure operations algebraically and studied the arising contact algebras that may yield spatial representations for all their models. In particular, this is the first time that non-distributive contact algebras are included and studied comprehensively as algebraic counterparts of mereotopologies. We showed that SPOCAs defined over Stonian p-ortholattices with $xCy \leftrightarrow x \not\leq y^\perp$ as contact are a good candidate for an ontologically coherent region-based theory of space. In fact, these are the least constrained algebraic structures that admit discrete C-extensional models amongst all of the algebraic theories satisfying the conditions of MT-representability which are at the same time necessary conditions for spatial representability. The other candidates for spatially representable contact algebras are BCAs, in particular its atomless extension RBCA, and the weaker GBCAs. These correspond to the logical theories RCC and GRCC known from the literature. While RCC models are C-extensional and always continuous, the models of GRCC can be discrete but are not C-extensional. We demonstrated that the

main difference between GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^\perp$ is whether complements are defined mereologically or topologically. Mereological complements require distributive contact algebras such as GBCA, BCA, or RBCA, while topological complements allow non-distributive contact algebra based on Stonian p-ortholattices. The remaining closure operations sum, intersections, and universal are in either case defined mereologically; topological sums require (C4) while a topological universal requires (Dis) or (C-Ext). As one of our key contributions, all mereological and topological closure operations are directly attributed to properties of the parthood lattice or the contact relation. Mereological complements manifest themselves in unique complementation in the algebraic counterparts while topological complements require (C5) which binds the contact relation to the orthocomplementation operation. Contact algebras with topological complements can be non-distributive, but are required to satisfy (Con), (C-Ext), and (C4). Thus the ontological choice of defining complements topologically is directly associated to many other, more implicit, ontological choices.

7.4 Outlook We have established in GBCAs and SPOCAs with (C5) two weakest, potentially spatially representable, theories that allow atoms and define all closures either mereologically or topologically. As natural next steps concrete topological embeddings theorems for these two classes of contact algebras need to be established analogue to the topological embeddings for BCAs [16]. For the SPOCAs with (C5), we know that non-representable models exist [48]. By extending the theory of SPOCAs with axioms that rule out some of the non-representable models, [48] is a first step towards such an embedding theorem.

Notes

1. Our notion of *spatial representability* deviates from standard topological representations in the sense that we are interested in whether all regions of an algebraic theory of mereotopology can be represented by adequately sized point sets so that notions such as contact (sharing a point), overlap (sharing a region), and complementation have intuitive spatial semantics. This understanding of spatial representations is similar to what are called a ‘faithful interpretations’ by [20; 35]. It is more stringent than the standard notion of topological representability of algebraic structures in pure mathematics.
2. ‘Algebraic counterpart’ refers to the class of contact algebras that can be constructed according to Theorem 3.1 Equally, a mereotopology whose models can be mapped to structures of a certain class of contact algebras is referred to as ‘logical counterpart’ of the class of contact algebras.
3. Orthocomplemented lattices have already been used in [4] as an algebraic theory of Clarke’s axiomatization of mereotopology.

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Appendix A Axioms for automated proofs

Axioms as used for the automated proofs in Prover9 (the notation has been slightly changed to make it more readable). $\&$, $|$, \rightarrow , and \leftrightarrow denote the logical connectives ‘and’, ‘or’, simple implication, and ‘if and only if’, respectively, while \wedge and \vee denote the lattice operations of meet and join. Universal closure is assumed throughout.

The theories $T_{OCA} = \{L2^\vee-L6^\vee, L2^\wedge-L4^\wedge, C0-C3, O1', O2', O3'\}$ and $T_{SPOCA} = T_{OCA} \cup \{PC1, PC2', PC2'', S\}$ axiomatize OCAs and SPOCAs.

Lattice: Standard axioms for commutativity, associativity, and absorption

$$\begin{array}{ll} (L2^\wedge) & x \wedge y = y \wedge x & (L2^\vee) & x \vee y = y \vee x \\ (L3^\wedge) & (x \wedge y) \wedge z = x \wedge (y \wedge z) & (L3^\vee) & (x \vee y) \vee z = x \vee (y \vee z) \\ (L4^\wedge) & x \vee (x \wedge y) = x & (L4^\vee) & x \wedge (x \vee y) = x \end{array}$$

Boundedness: Existence of a null and one (universal) element

$$(L5^\vee) \quad 0 \vee x = x \qquad (L6^\vee) \quad 1 \vee x = 1$$

Orthocomplementation and pseudocomplementation

$$\begin{array}{ll} (O1') & x^{\perp\perp} = x & (PC1) & x \wedge (x \wedge y)^* = x \wedge y^* \\ (O2') & x \vee x^\perp = 1 & (PC2') & 0^* = 1 \\ (O3') & x \wedge y = (x^\perp \vee y^\perp)^\perp & (PC2'') & 1^* = 0 \end{array}$$

The Stone identity

$$(S) \quad (x \vee y)^{**} = x^{**} \vee y^{**}$$

Contact: Basic axioms of a weak contact algebra

$$\begin{array}{ll} (C0) & 0 - \mathbf{C}x & (C1) & x \neq 0 \rightarrow x \mathbf{C}x \\ (C2) & x \mathbf{C}y \rightarrow y \mathbf{C}x & (C3) & \forall z(x \wedge y = x \ \& \ z \mathbf{C}x) \rightarrow z \mathbf{C}y \end{array}$$

Mereological closures as defined in Section 5.1

$$\begin{array}{ll} (M-I) & x \wedge y \neq 0 \rightarrow (z \wedge (x \wedge y) = z \leftrightarrow (z \wedge x = z \ \& \ z \wedge y = z)) \\ (M-S) & z \wedge (x \vee y) \neq 0 \leftrightarrow (x \wedge z \neq 0 \mid y \wedge z \neq 0) \\ (M-C) & z \wedge x^\perp = 0 \leftrightarrow z \wedge x = z \\ (O-Ext) & \forall z(x \wedge z = 0 \leftrightarrow y \wedge z = 0) \leftrightarrow x = y \end{array}$$

Topological closures as defined in Section 5.2

$$\begin{array}{ll} (T-I) & x \wedge y \neq 0 \rightarrow (z \mathbf{C}(x \wedge y) \rightarrow (z \mathbf{C}x \ \& \ w \mathbf{C}y)) \\ (T-S) & x \mathbf{C}(y \vee z) \leftrightarrow x \mathbf{C}y \mid x \mathbf{C}z \\ (C4) & x \mathbf{C}(y \vee z) \rightarrow x \mathbf{C}y \mid x \mathbf{C}z \\ (C5) & z \wedge x^\perp = z \leftrightarrow \neg z \mathbf{C}x & \simeq (T-C) \\ (C5') & (x \neq 0 \mid z \neq 1 \mid (x \neq 1 \ \& \ z \neq 0)) \rightarrow (z \mathbf{C}x \leftrightarrow (z = x^\perp \mid z \wedge x^\perp \neq z)) \simeq (T-C') \\ (C-Ext) & \forall z(x \mathbf{C}z \leftrightarrow y \mathbf{C}z) \leftrightarrow x = y \end{array}$$

Other axioms of interest

$$\begin{array}{ll} (Dis) & x \neq 1 \rightarrow \exists y(y \neq 0 \ \& \ x - \mathbf{C}y) \\ (Int) & x - \mathbf{C}y \rightarrow \exists z(x - \mathbf{C}z \ \& \ y - \mathbf{C}z^\perp) \\ (Con) & x = 0 \mid x = 1 \mid x \mathbf{C}x^\perp \\ (\neg Con) & x - \mathbf{C}x^\perp \\ (Uni) & (x \wedge y = 0 \ \& \ x \vee y = 1 \ \& \ x \wedge z = 0 \ \& \ x \vee z = 1) \rightarrow y = z & \text{(Unicomplemented)} \\ (Atom) & \exists a(a \neq 0 \wedge \forall x(x = 0 \mid x = a \mid x \wedge a \neq x)) & \text{(Existence of an atom)} \end{array}$$

Appendix B Equivalence of algebraic axioms

Here we show that the axioms from Section 5 in a UCMT are equivalent to the algebraic versions thereof, i.e. the axioms used for the automated proofs as shown in Appendix A. We can mainly rely on Theorem 3.1 but have to show additionally that all cases involving the introduced null element 0 are properly covered. We first show the equivalence for the mereological axioms and then for the topological axioms.

B.1 Mereological axioms

$$\begin{aligned} (\text{M-I}_{UCMT}) \quad & \forall w [P(w, x \odot y) \leftrightarrow (P(w, x) \wedge P(w, y))] && \text{(Intersection)} \\ (\text{M-S}_{UCMT}) \quad & \forall w [O(w, x \oplus y) \leftrightarrow (O(w, x) \vee O(w, y))] && \text{(Sum)} \\ (\text{M-C}_{UCMT}) \quad & \forall w [O(w, \ominus x) \leftrightarrow \neg P(w, x)] && \text{(Complement)} \end{aligned}$$

$$\begin{aligned} (\text{M-I}) \quad & x \wedge y \neq 0 \rightarrow [\forall z [z \wedge (x \wedge y) = z \leftrightarrow (z \wedge x = z \ \& \ z \wedge y = z)]] \\ (\text{M-S}) \quad & z \wedge (x \vee y) \neq 0 \leftrightarrow (x \wedge z \neq 0 \mid y \wedge z \neq 0) \\ (\text{M-C}) \quad & z \wedge x^\perp = 0 \leftrightarrow z \wedge x = z \end{aligned}$$

Lemma B.1 Let T_{UCMT} be the theory of UCMT that satisfies (P.1)–(P.3), (C.1)–(C.3), (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP). Let T_{OCA} be the theory of orthocomplemented contact algebras as constructed in Theorem 3.1. Then:

1. $T_{UCMT} \models (\text{M-I}_{UCMT})$ iff $T_{OCA} \models (\text{M-I})$;
2. $T_{UCMT} \models (\text{M-S}_{UCMT})$ iff $T_{OCA} \cup (\text{O-Ext}) \models (\text{M-S})$;
3. $T_{UCMT} \models (\text{M-C}_{UCMT})$ iff $T_{OCA} \cup (\text{O-Ext}) \models (\text{M-C})$.

Proof Let us define $z = g(w)$ throughout.

1. Assume (M-I_{UCMT}) . By definition $P(a, b) \Leftrightarrow a \leq b$ and because of $a \leq b \Leftrightarrow a \wedge b = a$ we obtain $z \wedge (x \wedge y) = z$ iff $z \wedge x = z$ and $z \wedge y = z$ for all $x, y, z \neq 0$. If $x = 0$ or $y = 0$, then $x \wedge y = 0$ and (M-I) holds. Otherwise, if $z = 0$, $z \wedge (x \wedge y) = 0 = z$ and $z \wedge x = 0 = z$ and $z \wedge y = 0 = z$ and thus (M-I) also holds. If (M-I) then for all $x, y, z \neq 0$ (M-I_{UCMT}) follows from $P(a, b) \Leftrightarrow a \wedge b = a$.
2. Assume (M-S_{UCMT}) , then if O is extensional by (O-Ext) and by definition of O we obtain $O(w, x \oplus y) \Leftrightarrow z \wedge (x \vee y) \neq 0$ and thus also (M-S) for all $x, y, z \neq 0$. If $z = 0$, then $z \wedge (x \vee y) = 0$ and $x \wedge z = 0$ and $y \wedge z = 0$. Analogue if both $x = 0$ and $y = 0$. If only $x = 0$ then $z \wedge (x \vee y) = z \wedge y$ iff $y \wedge z = 0$ since $x \wedge z = 0$. Analogue for $y = 0$. Reversely, if (M-S) then for all $x, y, z \neq 0$ (M-S_{UCMT}) directly follows.
3. Note that (M-C) is equivalent to $z \wedge x^\perp \neq 0 \Leftrightarrow z \wedge x \neq z$.

Assume (M-C_{UCMT}) . Since we already established that the complementation operator \ominus must at least satisfy the properties of an orthocomplementation, we have $O(w, \ominus x) \Rightarrow z \wedge x^\perp \neq 0$ in the presence of (O-Ext) and $\neg P(w, x) \Rightarrow z \wedge x \neq z$ which covers all cases of (M-C) in which $x, z \notin \{0, 1\}$. The remaining cases of (M-C) are:

- (i) If $z = 0$, then $z \wedge x^\perp = 0$ and $z \wedge x = 0 = z$.
- (ii) If $z = 1$ then $1 \wedge x^\perp = x^\perp \neq 0$ unless $x = 1$ and $z \wedge x = x \neq 1$ unless $x = 1$. The case when $x = 1$ is covered by (4).
- (iii) If $x = 0$ then $z \wedge 0^\perp = z \neq 0$ unless $z = 0$ and $z \wedge x = 0 \neq 0$ unless $z = 0$. The case $z = 0$ has already been covered by (1).
- (iv) If $x = 1$ then $z \wedge 1^\perp = 0$ and $z \wedge 1 = z$.

Reversely, if (M-C) then for all $x, z \neq 0$ and $x \neq 1$ (M-C_{UCMT}) directly follows.

□

B.2 Topological axioms

(T- I_{UCMT})	$\forall w[C(w, x \odot y) \rightarrow (C(w, x) \wedge C(w, y))]$	(Intersection)
(T- S_{UCMT})	$\forall w[C(w, x \oplus y) \leftrightarrow (C(w, x) \vee C(w, y))]$	(Sum)
(T- C_{UCMT})	$\forall w[P(w, \ominus x) \leftrightarrow \neg C(w, x)]$	(Complement)
(T- C'_{UCMT})	$\forall w[PP(w, \ominus x) \leftrightarrow \neg C(w, x)]$	(Alternative complement)

(T-I)	$x \wedge y \neq 0 \rightarrow (zC(x \wedge y) \rightarrow (zCx \ \& \ wCy))$
(T-S)	$zC(x \vee y) \leftrightarrow zCx \mid zCy$
(C5)	$z \wedge x^\perp = z \leftrightarrow \neg zCx$
(C5')	$(x \neq 0 \mid z \neq 1 \mid (x \neq 1 \ \& \ z \neq 0)) \rightarrow (zCx \leftrightarrow (z = x^\perp \mid z \wedge x^\perp \neq z))$

Lemma B.2 Let T_{UCMT} be the theory of **UCMT** that satisfies (P.1)–(P.3), (C.1)–(C.3), (UCMT.1)–(UCMT.7) with the definitions (O), (U), (PP). Let T_{OCA} be the theory of orthocomplemented contact algebras as constructed in Theorem 3.1 Then:

1. $T_{UCMT} \models (T-I_{UCMT})$ iff $T_{OCA} \models (T-I)$;
2. $T_{UCMT} \models (T-S_{UCMT})$ iff $T_{OCA} \models (T-S)$;
3. $T_{UCMT} \models (T-C_{UCMT})$ iff $T_{OCA} \models (C5)$.
4. $T_{UCMT} \models (T-C'_{UCMT})$ iff $T_{OCA} \models (C5')$.

Proof Notice that we again define $z = g(w)$ throughout.

1. Assume (T- I_{UCMT}), then (T-I) for all $x, y, z \neq 0$. If $z = 0$, for all $v \neg C(w, v)$ and thus (T-I). Otherwise, if $x = 0$ (or $y = 0$), then $x \wedge y = 0$ and thus also (T-I).
Reversely, if (T-I) then for all $x, y, z \neq 0$ (T- I_{UCMT}) directly follows.
2. Assume (T- S_{UCMT}), then (T-S) for all $x, y, z \neq 0$. If $z = 0$ then for all $v \neg C(w, v)$ and thus (T-S) holds. The same if $x = 0$ and $y = 0$. If only $x = 0$ then $C(z, y)$ if and only if $C(z, y)$. The same if $y = 0$.
Reversely, if (T-S) then for all $x, y, z \neq 0$ (T- S_{UCMT}) directly follows.
3. Assume (T- C_{UCMT}). Since \ominus must at least satisfy the properties of an orthocomplementation $P(w, \ominus x) \Leftrightarrow z \wedge x^\perp = z$. The remaining cases are:
 - (i) If $z = 0$ then $z \wedge x^\perp = 0 = z$ and $\neg C(0, x)$.
 - (ii) If $z = 1$ then $1 \wedge x^\perp = x^\perp \neq z$ unless $x = 0$ and $C(w, x)$ unless $x = 0$. The case when $x = 0$ is covered by (3).
 - (iii) If $x = 0$ then $z \wedge 0^\perp = z$ and $\neg C(w, 0)$.
 - (iv) If $x = 1$ then $z \wedge 1^\perp = 0 \neq z$ unless $z = 0$ and $C(z, 1)$ unless $z = 0$. The case when $z = 0$ is covered by (1).
Reversely, if (T-C) then for all $x, z \neq 0$ and $x \neq 1$ (T- C_{UCMT}) directly follows.
4. We can rewrite (T- C'_{UCMT}) as $\forall w[\neg PP(w, \ominus x) \leftrightarrow C(w, x)]$. Assume (T- C'_{UCMT}). Then $\neg PP(w, \ominus x) \Leftrightarrow z = x^\perp$ or $z \wedge x^\perp \neq z$ since \ominus must at least satisfy the properties of an orthocomplementation. Then for all $x, z \notin \{0, 1\}$ (C5') holds. Trivially, (C5') holds if $x = 0$ or $z = 1$ or $x = 1$ and $z = 0$. The remaining cases are:
 - (i) If $z = 0$ then $\neg C(0, x)$ and $0 \neq x^\perp$ unless $x = 1$ and $0 \wedge x^\perp = 0 = z$. The case $x = 1$ is covered by the precondition of (C5').
 - (ii) If $x = 1$ then $C(z, 1)$ unless $z = 0$ and $z \neq 1^\perp$ unless $z = 0$ and $z \wedge 1^\perp = 0 \neq z$ unless $z = 0$. The case $z = 0$ is covered by the precondition of (C5').
Reversely, if (T-C') then for all $x, z \neq 0$ and $x \neq 1$ (T- C'_{UCMT}) directly follows.

□