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Problems with Fields as Functions

The field is a geospatial model for representing measurements or predictions of continuous spatiotemporal phenomena. As such, it aims to give a value for any position in a region or across a span of time. The mathematical definition of a function is a mapping of one set, the domain, onto another set, the co-domain (or range), and so one obvious way to model a field mathematically is to view it as a function. The definition of a function is fairly restrictive: for every point in the domain, one and only one value can result. Additionally, since the phenomena described by the field is spatially continuous, this places restrictions on what types of sets the domain can be. The domain of a field corresponds to spatial or temporal (or both) reality, and it is "continous" (which might have differing meanings in different situations). It's worth examining what consequences can arise from this conception of a field as a function over a continuous domain and exploring what possibilities this model will allow.

First, a word on terminology. The term field as used here refers to a geographic or spatiotemporal field (or possibly a spatial field or a temporal field). It is a close cognate to the fields used by physicists and engineers. A similar term is that of coverage. Data structures which implement this model directly (at least to some degree) are the raster or the TIN (Worboys & Duckham) in that they aim to give a result for any point within them. It has no relation to the term field as it is used in computing and databases where it refers to an attribute or variable associated with a record. It also doesn't refer to the mathematical concept of field which is a set with two group forming operators (e.g. integer, real, or complex numbers).

The field model can be formulated in a number of ways, but one very common (and useful) way is to view it as a function from position to attribute (Coulcelis, Kemp, Worboys & Duckham). As a function, the field has a domain, the set of all possible values for which the field is defined, and a range, the set of all possible values which could result from the field. This translates very naturally to the underlying physical phenomenon being described, and this model is useful for analysis of the phenomenon since the mathematical tools of physics, meteorology, and spatial analysis are based on mathematical functions. But there may be situations where what results from a field is not single valued, and it may be that over a consistent domain, points may exist where the field has no value associated. This may have consequences for whether the field is considered continuous.

The term continuous has a number of definitions. Mathematically, a function is continuous if as the range of a independent variable shrinks so does the corresponding range of the dependent variable (Adams & Franzosa). In topology, this is generalized as a continuous function being a one where open sets have open preimages (the set of all points which map into the open set) (Adams & Franzosa). However, both of these definitions are applied to functions, and we are talking about the continuous domain of a field, that is a continuous set.

The mathematical concept of continuum is often used to understand what is described as continuous as applied to a region or set (Bell). The continuum is the prototype of the real number line (Adams & Franzosa). It is has no gaps, and put in terminology of sets, it is connected, convex, and dense. The continuum is connected which means that a path can be found between any two points in the set, and in the single dimension of the continuum, this is equivalent both to its convexity and density (diagram). But in two dimensions, sets can exist which are connected but not convex (diagram) although convexity does imply connectedness. The continuum also excludes discrete sets, which are used whenever a digital implementation of a field is used.

But this definition of field as a function does not capture the full range of geographic information. Situations exits where the behavior of the field as a function fails to fully model reality can be constructed where multiple values at a position are needed, and the model of the field as being defined over a continuous domain is overly restrictive for modeling things like discontiguous regions. The next section of this talk deals with such examples.

One prototypical example of a field is that of elevation, often implemented as a digital elevation model (DEM). But at some point within the field there might be a cliff with an overhang. If elevation is defined as the interface between the ground and the atmosphere, the field ceases to be a simple function. (Diagram.) Yes, the field can be defined to be the elevation of the highest ground point at a position, but this loses information about the terrain. For visualization or hydrological analysis, this information can be important, as is having a data model which retains it. In these situations, the elevation field is best modeled mathematically as a relation.

Another example of multivalued fields...

If a field of land surface temperature is defined over Hawaii, because no value exists for points which are not over land, the domain over which the field is defined is disconnected because of the structure of islands embedded in an ocean. We could propose a different definition of the field: either water surface temperature could be used as a stand in for land surface temperature to make the domain continuous, or each island could be represented as one of over a hundred a separate isolated fields. A field could be considered a mathematical relation instead of a function, and then there is no problem with having points where no value results. These possible solutions each have their own problems. Water surface temperature has a different meaning than land surface temperature, and different measurement methods may be required. And aside from the inconvenience of having to deal with more field objects, the inability to combine disjoint fields can hamper any analysis which needs to work across the fields.

When a field is defined over a continuous domain, this is sometimes equated with there being "no gaps" in the domain. This idea of no gaps deserves some examination because it can lead to something more restrictive than is intended. Certainly, it means that between any two points in the domain, there is also another point in the domain, and continuing in this way (between those two points, and between those two points...), we come to the idea of continuum. However, because we are working in two (or more) dimensions, the idea of "between" may not be as well-defined as we might like. If between is intended to convey everything along some chosen path, then the idea of no gaps becomes equivalent to the idea of connected (i.e. there is some path between two points). But if between is intended to convey everything along the straight-line path between points, this is the definition of convex. While the rectangular regions of rasters and the convex hulls which define regions where interpolation can be performed meet this more restrictive definition, arbitrary regions do not, and the realization that not all points "between" two points in the domain will also be in the domain may be important.

When we define data structures which implement fields in a computer we are setting up a relation between position and value, but the domain of the set cannot be a connected set. If the field is defined over integers, what is being used is a discrete set of isolated points, and no path exists between any two points. The situation is no better when floating point number representations are used. There are alternative topologies which can be imposed over discrete sets which do deal with connectedness, but these digital line and digital plane topologies are not currently used for in computing (Adams & Franzosa).

In order to do mathematical analysis directly on fields, the idea of field operators has been proposed. The idea is to have a set of operators which are closed over fields; that is operators which take one or more fields and use them to return a new field. Much of the motivation for this has been to mirror what can be done with map algebra on raster data, but the hope is that a full complement of arithmetic, statistical, and logical operators can be developed for the broader and more abstract field model and can be used directly for analysis of fields. We will see that while the idea of the field as a function over a continuous domain isn't incompatible with the idea of field operators, without particular care, inconsistencies can result.

One such operator proposed is the predicate operator. This operator can be formulated in a number of ways, but the aim is to produce a new field based on a proposition evaluated as either true or false. One proposed definition is to have the resulting field be defined with the original attributes if and only if the proposition is true; where the proposition is false, the field is undefined. Points in the original domain where the proposition is false are no longer part of the resulting domain, and since nothing about the predicates require that the remaining region be connected, the domain of the resulting field need not be connected. (diagram).

Another possible way to work with the results of field predicates is as a field of Boolean values which are defined at every point in the original domain: true where the underlying proposition is true, and false where it is false. Since every point in the original domain is defined, if the original domain is connected, the resulting domain is also connected. It may appear we have solved the problem of disconnected domains with predicate operators.

We can take this idea a step further and define a binary Boolean-and field operator which is defined as true when the Boolean and of the corresponding underlying points is true and false when false (diagram). But if the two fields don't coincide completely, the possibility exists for a disconnected resulting domain. For example, consider one region of an operand field which is false and disjoint from the other operand field. Because false combined any other value is false for an and operator, we know the resulting field will be defined as false over this region even though the second operand field isn't defined there. Because this region surrounded by true values and disjoint from the other operand field, there is a region which overlaps the operand field for which we do not know the value; the field is undefined in this region. And finally, in the region where the two operand fields do overlap, the values are known and therefore the field is defined. In this situation, the resulting field has a disconnected domain. One possible solution is to only define such an operator (and similar operators) only over fields with completely coincident domains. While this does solve the problem for this case, it restricts situations where analysis can be done to only where regions overlap completely.

So what can be done? In many situations, treating a field as a function over a connected region is necessary. When further analysis is primarily mathematically based, and that mathematics is based on simple continuous functions, having a field which is defined over a consistent and connected region. In other situations, a field, while being remaining a spatial relation, cannot be defined as a function since points in its domain might have no value and other points have multiple values. Whether a field is a function or not has impacts on how that field can be used for further analysis.

It may be that closed field operators will need to be defined on fields of completely coincident regions and only on connected (or possibly even convex) domains. The exceptions and complexities which arise from defining the operators otherwise may become too difficult to manage.

It may be that the most general definition of a field is most appropriately not a function, but a relation. This has the advantage of describing a wider variety of spatial situations, but it has added complexity and allows for fewer assumptions to be made.

More likely is that geographical analysis will require both: simplistic functional fields for mathematical analysis and richer (but more complicated) relational fields for analysis which goes beyond the mathematical. Which fits better into an intuitive model of a field remains to be seen.

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